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The comparison between the Leonard II theorem and Putzer algorithm for solving some linear differential equations

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ABSTRACT

Homogeneous Linear ordinary differential equations are a class of differential equations having many important uses in engineering and sciences. This research discusses a comparison between Via-Leonard II theorem and Putzer algorithm theorem for solving some homogeneous linear ordinary differential equations. This study shows the existence of unique solution, and the calculations are simpler and faster than in traditional techniques.

Keywords: Via-Leonard II theorem; Putzer algorithm theorem; homogeneous linear and nonlinear ordinary differential equations.

1. Introduction

The Putzer Algorithm is a method for analytically evaluating matrix exponentials using only eigenvalues and components in the solution of a relatively simple linear system. It is particularly useful for matrices that cannot be diagonalized because it avoids the use of Jordan canonical form (for more details see [1] and [2]). Many researches have conducted so far about some application of Leonard II theorem (for details see [3], [4] and [5]). The putzer algorithm is not a method to solve differential equations` but also it used to solve impulsive differential equations (for more details see [6]). According to these studies, this paper will show the comparison between these methods on both linear and nonlinear differential equations.

2. Homogeneous linear ordinary differential equations

Theorem 2.1 (Putzer algorithm**)** [1]: Let A be an $n \times n$ matrix with egivenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then

$$e^{At} = \sum_{k=1}^{n} r_k(t) P_{k-1}$$

and

$$P(\lambda) = \det(\lambda I - A) = (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$$

Where

$$P_0 = I$$

and

$$P_k = (A - \lambda I)P_{k-1} \text{ , for} \geq 1 \text{ ,}$$

let

$$\dot{r_1}(t) = \lambda_1 r_1(t), r_1(0) = 1, \text{ for } k > 1$$

and

$$\dot{r_k}(t) = \lambda_k r_k(t) + r_{k-1}(t),\ r_k(0) = 0.$$

Proof:

Let

$$\emptyset(\mathrm{t}) = \sum_{k=1}^{n} r_k(t)P_{k-1},$$

$$\emptyset(0) = \sum_{k=1}^{n} r_{(k)}(0)P_{k-1}$$

$$\emptyset(0) = r_1(0)P_0 + r_2(0)P_1 + \cdots + r_n(0)P_{n-1}$$

$$= 1 * P_0 + 0 * P_2 + \cdots + 0 * P_{n-1} = P_0 = I$$

$$\dot{\emptyset}(t) = \sum_{k=1}^{n} \dot{r_k}(t)P_{k-1} = \dot{r_1}(t)P_0 + \sum_{k=2}^{n} \dot{r_k}(t)P_{k-1}$$

$$= \lambda_1 r_1(t)P_0 + \sum_{k=2}^{n} [\lambda_k r_k(t) + r_{k-1}(t)]P_{k-1}$$

$$= r_1(t)[\lambda_1 P_0] + \sum_{k=2}^{n} r_k(t)[\lambda_k P_{k-1}] + \sum_{k=2}^{n} r_{k-1}(t)P_{k-1}$$

$$= r_1(t)[AP_0 - P_1] + \sum_{k=2}^{n} r_k(t)[AP_{k-1} - P_k] + \sum_{k=2}^{n} r_{k-1}(t)P_{k-1}$$

$$= A[r_1(t)P_0 + \sum_{k=2}^{n} r_k(t)P_{k-1}] + r_n(t)P_k$$

$$= A\sum_{k=1}^{n} r_k(t)P_{k-1} + 0$$
$$= A\emptyset(t)\ ; \quad \emptyset(t) = e^{At}$$

Assume that A is an $n \times n$ matrix with characteristic polynomial function $\det(\lambda I - A) = (\lambda - \lambda_0)^n$, then the functions $r_k(t)$ for which

$$e^{At} = \sum_{k=1}^{n} r_k(t)(A - \lambda_0 I)^{k-1}$$

To show that: let $P(\lambda)$ of given by as

$$P(\lambda) = \det(A - \lambda_0 I)^{k-1} = (\lambda - \lambda_0)(\lambda - \lambda_0) \ldots (\lambda - \lambda_0);\ \lambda = \lambda_0, \lambda_0, \ldots, P_0 = I$$

and

$$P_k = (A - \lambda_0 I)P_{k-1} \text{ , for} \geq 1 \text{ ,}$$

let

$$\dot{r_1}(t) = \lambda_1 r_1, r_1(0) = 1, \lambda = \lambda_0,$$

$$\frac{dr_1}{dt} = \lambda_0 r_1(t) \Longrightarrow \frac{dr_1}{r_1} = \lambda_0 dt \Longrightarrow ln r_1 = \lambda_0 t + c \Longrightarrow r_1 = e^{\lambda_0 t}.$$

$$\dot{r}_2(t) = \lambda_2 r_2(t) + r_1(t),$$

and

$$r_2(0) = 0, \dot{r}_2(t) = \lambda_0 r_2(t) + e^{\lambda_0 t}$$

Multiplying throughout by the integrating factor

$$e^{-\lambda_0 t}\ , \dot{r}_2(t)e^{-\lambda_0 t} - \lambda_0 r_2(t)e^{-\lambda_0 t} = 1,$$

$$\frac{d}{dt}\left(e^{-\lambda_0 t} r_2(t)\right) = 1 \Longrightarrow e^{-\lambda_0 t} r_2(t) = t + c \ \Longrightarrow r_2(t) = te^{\lambda_0 t}$$

Theorem 2.2 [2](Leonard I). Let $A \in R^{n\times n}$ be a constant matrix with characteristic polynomial

$$p(\lambda) = det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0;$$

Then, $\Phi(t) = e^{At}$ is the unique solution to the nth-order matrix differential equation.

$$\Phi^{(n)}(t) + c_{n-1}\Phi^{(n-1)}(t) + \cdots + c_1\Phi^{(1)}(t) + c_0\Phi(t) = 0$$

With initial conditions

$$\Phi(0) = \mathrm{I}\, , \Phi^{(1)}(0) = A\, , \ldots,\ \Phi^{(n-2)}(0) = A^{n-2}\, ,\ \Phi^{(n-1)}(0) = A^{n-1}.$$

Proof: By [2] .

Theorem 2.3 (Leonard II) [2] . Let $A \in R^{n\times n}$ be a constant matrix with characteristic polynomial $p(\lambda) = det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$. Then

$$e^{At} = x_0(t)I + x_1(t)A + x_2(t)A^2 + \cdots + x_{n-1}(t)A^{n-1},$$

Where $x_k(t), k \in Z_n$ are the solutions to the nth-order scalar ODEs

$$\mathrm{x}^{(n)}(t) + c_{n-1}\mathrm{x}^{(n-1)}(t) + \cdots + c_1\mathrm{x}^{(1)}(t) + c_0\mathrm{x}(t) = 0,$$

Satisfying the initial conditions

$$x_k^{(j)}(0) = \delta_{j-k}\ for\ j,k \in z_n \left(x_k^{(0)}(t) = x_k(t)\right),$$

Proof: Follows from [2].

Example 2.1. Suppose that

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$$

Which has the eigenvalues $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$. Begin by assuming distinct eigenvalues for A. The general solution to the third-order linear homogeneous ordinary differential equations.

$$x^{(3)}(t) + c_2x^{(2)}(t) + c_1x^{(1)}(t) + c_0x(t) = 0$$

With characteristic roots 1,2,3 are

$$x(t) = a_0e^t + a_1e^{2t} + a_2e^{3t}$$

$$x^{(1)}(t) = a_0e^t + 2a_1e^{2t} + 3a_2e^{3t}$$

$$x^{(2)}(t) = a_0e^t + 4a_1e^{2t} + 9a_2e^{3t}\ .$$

Where, a_0, a_1 and a_3 are constants, it is easy to determine them by using the initial conditions, for the initial conditions $x(0) = 1,\ x^{(1)}(0) = 0, x^2(0) = 0$, we obtain the linear systems.

$$\begin{aligned} a_0 + a_1 + a_2 &= 1, \\ a_0 + 2a_1 + 3a_2 &= 0, \\ a_0 + 4a_1 + 9a_2 &= 0, \end{aligned}$$

After solving this system, we obtain $a_0 = 3, a_1 = -3, a_2 = 1$. Thus, the particular solution in this case is $x_0(t) = 3e^t - 3e^{2t} + e^{3t}$.

Now, by changing the initial conditions to $x(0) = 0$, $x^{(1)}(0) = 1, x^2(0) = 0$, then we have the following system.

$$\begin{aligned} a_0 + a_1 + a_2 &= 0, \\ a_0 + 2a_1 + 3a_2 &= 1, \\ a_0 + 4a_1 + 9a_2 &= 0, \end{aligned}$$

by solving the system we get $a_0 = \frac{-5}{2}, a_1 = 4, a_2 = \frac{-3}{2}$. Thus, the solution in this case is $x_1(t) = -\frac{5}{2}e^t + 4e^{2t} - \frac{3}{2}e^{3t}$

If instead the initial conditions are given as $x(0) = 0$, $x^{(1)}(0) = 0, x^2(0) = 1$, we have the linear system of equations.

$$\begin{aligned} a_0 + a_1 + a_2 &= 0, \\ a_0 + 2a_1 + 3a_2 &= 0, \\ a_0 + 4a_1 + 9a_2 &= 1, \end{aligned}$$

Which has the solution $a_0 = \frac{1}{2}, a_1 = -1, a_2 = \frac{1}{2}$. Thus,

$x_2(t) = \frac{1}{2}e^t - e^{2t} + \frac{1}{2}e^{3t}$.

Via Leonard II we must have:

$$e^{At} = x_0(t)I + x_1(t)A + x_2(t)A^2$$

$$e^{At} = \begin{pmatrix} -e^t + 2e^{2t} & -2e^t + 2e^{2t} & -2e^t + 2e^{2t} \\ -e^{2t} + e^{3t} & -e^{2t} + 2e^{3t} & -e^{2t} + e^{3t} \\ e^t - e^{3t} & 2e^t - 2e^{3t} & 2e^t - e^{3t} \end{pmatrix}.$$

Example 2.2. Find e^{At} using the putzer algorithm where

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$$

The solution:

$$e^{At} = \sum_{k=1}^{2} r_k(t)p_{k-1} = r_1(t)p_0 + r_2(t)p_1,$$

$$p_0 = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

And

$p_1 = (A - \lambda_1 I)p_0$, $p_2 = (A - \lambda_2 I)p_1$, $p(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & -2 & -2 \\ -1 & \lambda - 4 & -1 \\ 2 & 4 & \lambda + 1 \end{vmatrix} = 0$, $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$

$$p_1 = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ -2 & -4 & -2 \end{pmatrix} \quad \text{and} \quad p_2 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 4 & 2 \\ -2 & -4 & -2 \end{pmatrix}$$

$\frac{dr_1(t)}{dt} = \lambda_1 r_1(t)$

and

$r_1(0) = 1 \Longrightarrow \frac{dr_1(t)}{r_1} = dt \Longrightarrow r_1(t) = e^t$,

$\frac{dr_2}{dt} = \lambda_2 r_2(t) + r_1$ and $r_2(0) = 0$, $\frac{dr_2}{dt} - 2r_2(t) = e^{-t}$

Multiplying throughout by the integrating factor e^{-2t},

$\frac{dr_2}{dt} e^{-2t} - 2r_2(t)e^{-2t} = e^{-t} \Longrightarrow \frac{d(r_2(t)e^{-2t})}{dt} = e^t \Longrightarrow r_2(t)e^{-2t} = -e^{-t} + c \Longrightarrow r_2(t) = -e^t + e^{-2t}$, $\frac{dr_3}{dt} = \lambda_3 r_3 + r_2$

and

$r_2(0) = 0$, $\frac{dr_3}{dt} - 3r_3(t) = e^{-t} + e^{2t}$

Multiplying throughout by the integrating factor e^{-3t},

$\frac{dr_3}{dt} e^{-3t} - 3r_2(t)e^{-3t} = e^{-t} \Longrightarrow \frac{d(r_3(t)e^{-3t})}{dt} = -e^{-t} - e^{-2t}$

$\Longrightarrow r_3(t)e^{-3t} = -e^{-t} - \frac{1}{2}e^{-2t} + c \Longrightarrow r_3(t) = -\frac{1}{2}e^t + \frac{1}{2}e^{2t} + \frac{1}{2}e^{3t}$,

$$e^{At} = p_0 r_1 + p_1 r_2 + p_2 r_3$$

$$e^{At} = \begin{pmatrix} -e^t + 2e^{2t} & -2e^t + 2e^{2t} & -2e^t + 2e^{2t} \\ -e^{2t} + e^{3t} & -e^{2t} + 2e^{3t} & -e^{2t} + e^{3t} \\ e^t - e^{3t} & 2e^t - 2e^{3t} & 2e^t - e^{3t} \end{pmatrix}.$$

Example 2.3. Suppose that

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \in R^{3\times 3}$$

Which has the eigenvalues $\lambda_1 = 2, \lambda_2 = 2, \lambda_3 = 2$. Begin by assuming equal eigenvalues for A. The general, solution to the third-order homogeneous ordinary differential equations is such that

$$x^{(3)}(t) + c_2 x^{(2)}(t) + c_1 x^{(1)}(t) + c_0 x(t) = 0$$

With characteristic roots 2.

$$x(t) = (a_0 + a_1 t + t^2)e^{2t}$$

$$x^{(1)}(t) = 2a_0 e^{2t} + a_1 e^{2t}$$

$x^{(2)}(t) = 4a_0 e^{2t} + 4a_1 e^{2t} + 2a_2 e^{2t}$.

For the initial conditions $x(0) = 1$, $x^{(1)}(0) = 0, x^{(2)}(0) = 0$, we obtain the linear system.

$$a_0 = 1$$
$$2a_0 + a_1 = 0$$
$$4a_0 + 4a_1 + 2a_2 = 0,$$

the solution yields $a_0 = 1, a_1 = -2, a_2 = 2$. Thus, the solution in this case is $x_0(t) = e^{2t} - 2te^{2t} + 2t^2 e^{2t}$.

Now, by changing the initial conditions by $x(0) = 0$, $x^{(1)}(0) = 1, x^{(2)}(0) = 0$, we obtain the linear system.

$$a_0 = 0$$
$$2a_0 + a_1 = 1$$
$$4a_0 + 4a_1 + 2a_2 = 0,$$

by solving the system, we obtain $a_0 = 0, a_1 = 1, a_2 = -2$. Thus, the solution can be written as $x_1(t) = te^{2t} - 2t^2 e^{2t}$. Now, if instead the initial conditions $x(0) = 0$, $x^{(1)}(0) = 0, x^{(2)}(0) = 1$, we obtain the linear system.

$$a_0 = 0$$
$$2a_0 + a_1 = 0$$
$$4a_0 + 4a_1 + 2a_2 = 1,$$

the solution yields $a_0 = 0$, $a_1 = 0$, $a_2 = \frac{1}{2}$. Thus, the solution in this case is

$x_2(t) = \frac{1}{2} t^2 e^{2t}$.

We have:

$$e^{At} = x_0(t)I + x_1(t)A + x_2(t)A^2$$

$$e^{At} = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix} e^{2t}.$$

Example 2.4. Find e^{At} using the putzer algorithm if

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

The solution:

$$e^{At} = \sum_{k=1}^{3} r_k(t) p_{k-1}$$
$$= r_1(t)p_0 + r_2(t)p_1 + r_3(t)p_2,$$

$$p_0 = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and $p_1 = (A - \lambda_1 I)p_0$, $p_2 = (A - \lambda_2 I)p_1$,

$$p(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 1 & \lambda - 2 \end{vmatrix} = 0,$$

$\lambda_1 = 2, \lambda_2 = 2 , \lambda_3 = 2.$ $p_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and

$$p_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\frac{dr_1(t)}{dt} = \lambda_1 r_1(t)$ and $r_1(0) = 1 \Longrightarrow \frac{dr_1(t)}{r_1} = dt \Longrightarrow r_1(t) = e^{2t}$,

$\frac{dr_2}{dt} = \lambda_2 r_2(t) + r_1$ and $r_2(0) = 0$, $\frac{dr_2}{dt} - 2r_2(t) = e^{2t}$

Multiplying throughout by the integrating factor e^{-2t},

$\frac{dr_2}{dt} e^{-2t} - 2r_2(t)e^{-2t} = 1 \Longrightarrow \frac{d(r_2(t)e^{-2t})}{dt} = 1 \Longrightarrow r_2(t)e^{-2t} = t + c \Longrightarrow r_2(t) = te^{2t}, \frac{dr_3}{dt} = \lambda_3 r_3 + r_2$

and

$r_2(0) = 0$, $\frac{dr_3}{dt} - 2r_3(t) = te^{2t}$

Multiplying throughout by the integrating factor e^{-2t},

$\frac{dr_3}{dt} e^{-2t} - 2r_2(t)e^{-2t} = t \Longrightarrow \frac{d(r_3(t)e^{-2t})}{dt} = t \Longrightarrow r_3(t)e^{-2t} = \frac{t^2}{2} + c \Longrightarrow r_3(t) = \frac{1}{2}t^2 e^{2t},$ $e^{At} = p_0 r_1 + p_1 r_2 + p_2 r_3$

$$e^{At} = e^{2t} \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix}.$$

Example 2.5. Suppose that

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \in R^{4\times 4}$$

Which has the eigenvalues $\lambda_1 = i, \lambda_2 = i, \lambda_3 = -i , \lambda_4 = -i$. Begin by assuming imaginary eigenvalues for A. The general, solution to the fourth order homogeneous ordinary differential equations is given by.

$$x^{(4)}(t) + c_3 x^{(3)}(t) + c_2 x^{(2)}(t) + c_1 x^{(1)}(t) + c_0 x(t) = 0$$

With characteristic roots $\pm i, \pm i$

$$\boldsymbol{x(t) = (a_0 + a_1 t)e^{it} + (a_2 + a_3 t)e^{-it}}$$
$$\boldsymbol{x^{(1)}(t) = ia_0 e^{it} + a_1 e^{it} - ia_2 e^{-it} + a_3 e^{-it} - ita_3 e^{-it} + ita_1 e^{it}}$$
$$\boldsymbol{x^{(2)}(t) = -a_0 e^{it} + 2ia_1 e^{it} - a_2 e^{-it} - 2ia_3 e^{-it} - ta_3 e^{-it} - ta_1 e^{it}}$$
$$\boldsymbol{x^{(3)}(t) = -ia_0 e^{it} - 3a_1 e^{it} + ia_2 e^{-it} - 3a_3 e^{-it} + ita_3 e^{-it} - ita_1 e^{it}}$$

For the initial conditions $x(0) = 1, x^{(1)}(0) = 0, x^{(2)}(0) = 0$, $x^{(3)}(0) = 0$, we get the following system.

$$\begin{aligned} a_0 + a_2 &= 1 \\ ia_0 + a_1 - ia_2 + a_3 &= 0 \\ -a_0 + 2ia_1 - a_2 - 2ia_3 &= 0 \\ -ia_0 - 3a_1 + ia_2 - 3a_3 &= 0 \end{aligned}$$

the solution yields $a_0 = \frac{1}{2}, a_1 = -\frac{1}{4}i, a_2 = \frac{1}{2}, a_3 = \frac{1}{4}i.$ Thus, the solution in this case is $x_0(t) = \frac{1}{2}e^{it} - \frac{1}{4}ite^{it} + \frac{1}{2}e^{-it} + \frac{1}{4}ite^{-it}$.

If instead the initial conditions $x(0) = 0, x^{(1)}(0) = 1, x^{(2)}(0) = 0 , x^{(3)}(0) = 0$ we have the linear system of equations.

$$\begin{aligned} a_0 + a_2 &= 0 \\ ia_0 + a_1 - ia_2 + a_3 &= 1 \\ -a_0 + 2ia_1 - a_2 - 2ia_3 &= 0 \\ -ia_0 - 3a_1 + ia_2 - 3a_3 &= 0 \end{aligned}$$

the solution yields $a_0 = -\frac{3}{4}i, a_1 = -\frac{1}{4}, a_2 = \frac{3}{4}i, a_3 = -\frac{1}{4}$ Thus, the solution in this case is $x_1(t) = -\frac{3}{4}ie^{it} - \frac{1}{4}te^{it} + \frac{3}{4}ie^{-it} - \frac{1}{4}te^{-it}$.

If instead the initial conditions $x(0) = 0, x^{(1)}(0) = 0, x^2(0) = 1 , x^{(3)} = 0.$ we obtain the following system.

$$\begin{aligned} a_0 + a_2 &= 0 \\ ia_0 + a_1 - ia_2 + a_3 &= 0 \\ -a_0 + 2ia_1 - a_2 - 2ia_3 &= 1 \\ -ia_0 - 3a_1 + ia_2 - 3a_3 &= 0 \end{aligned}$$

the solution yields $a_0 = 0 , a_1 = -\frac{1}{4}i, a_2 = 0, a_3 = \frac{1}{4}i.$ Thus, the solution in this case is $x_2(t) = -\frac{1}{4}ite^{it} + \frac{1}{4}ite^{-it}$.

$$\begin{aligned} a_0 + a_2 &= 0 \\ ia_0 + a_1 - ia_2 + a_3 &= 0 \\ -a_0 + 2ia_1 - a_2 - 2ia_3 &= 0 \\ -ia_0 - 3a_1 + ia_2 - 3a_3 &= 1 \end{aligned}$$

conditions $x(0)=0$, $x^{(1)}(0)=0$, $x^{2}(0)=0$,$x^{(3)}=1$ we have the linear system of equations. The solution yields $a_0=-\frac{1}{4}i$,$a_1=-\frac{1}{4}$,$a_2=\frac{1}{4}i$. $a_3=-\frac{1}{4}$.Thus, the solution in this case is

$$x_3(t)=-\tfrac{1}{4}ie^{it}-\tfrac{1}{4}te^{it}+\tfrac{1}{4}ie^{-it}-\tfrac{1}{4}te^{-it}.$$

We follow same procedure, to obtain:

$$e^{At}=x_0(t)I+x_1(t)A+x_2(t)A^2+x_3(t)A^3=$$

$$\begin{pmatrix} \cos(t) & \sin(t) & 0 & 0 \\ -\sin(t) & \cos(t) & 0 & 0 \\ \frac{t}{2}\sin(t) & \frac{1}{2}\sin(t)-\frac{t}{2}\cos(t) & \cos(t) & -\sin(t) \\ \frac{1}{2}\sin(t)+\frac{t}{2}\cos(t) & \frac{t}{2}\sin(t) & -\sin(t) & \cos(t) \end{pmatrix}.$$

Example 2.6. Find e^{At} using the putzer algorithm if

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

The solution:

$$e^{At}=\sum_{k=1}^{4} r_k(t)p_{k-1}=r_1(t)p_0+r_2(t)p_1+r_3(t)p_2+r_4(t)p_3,$$

$$p_0=I=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

And $p_1=(A-\lambda_1 I)p_0$, $p_2=(A-\lambda_2 I)p_1$, $p_3=(A-\lambda_3 I)p_2$,

$$p(\lambda)=\det(\lambda I-A)=\begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{vmatrix}=0\ ,\ \lambda_1=$$
$i,\lambda_2=i$,$\lambda_3=-i,\lambda_4=-i.$,

$$p_1=\begin{pmatrix} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 1 & 0 & -1 & -i \end{pmatrix},\quad p_2=\begin{pmatrix} -2 & -2i & 0 & 0 \\ 2i & -2 & 0 & 0 \\ 1 & 0 & -2 & -2i \\ -2i & 1 & 2i & -2 \end{pmatrix},$$

$$p_3=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \end{pmatrix}.$$

$$\frac{dr_1(t)}{dt}=\lambda_1 r_1(t)\ \text{and}\ r_1(0)=1\Longrightarrow\frac{dr_1(t)}{r_1}=dt\Longrightarrow r_1(t)=e^{it},$$

$$\frac{dr_2}{dt}=\lambda_2 r_2(t)+r_1\ \text{and}\ r_2(0)=0\ ,\ \frac{dr_2}{dt}-ir_2(t)=e^{it}$$

Multiplying throughout by the integrating factor e^{-it},

$$\frac{dr_2}{dt}e^{-it}-ir_2(t)e^{-it}=1\Longrightarrow\frac{d(r_2(t)e^{-it})}{dt}=1\Longrightarrow r_2(t)e^{-t}=t+c\Longrightarrow r_2(t)=te^{it},$$

$$\frac{dr_3}{dt}=\lambda_3 r_3+r_2\ \text{and}\ r_3(0)=0\ ,\ \frac{dr_3}{dt}+ir_3(t)=te^{it}$$

Multiply throughout by the integrating factor e^{it},

$$\frac{dr_3}{dt}e^{it}+ir_3(t)e^{it}=te^{2it}\Longrightarrow\frac{d(r_3(t)e^{it})}{dt}=te^{2it}$$

$$\Longrightarrow r_3(t)e^{it}=-\tfrac{1}{2}ite^{2it}+\tfrac{1}{4}e^{2it}+c\Longrightarrow r_3(t)=\tfrac{1}{4}e^{it}-\tfrac{1}{2}ite^{it}-\tfrac{1}{4}e^{-it},$$

$$\frac{dr_4}{dt}=\lambda_4 r_4+r_3\ \text{and}\ r_4(0)=0\ \frac{dr_4}{dt}+ir_4(t)=\tfrac{1}{4}e^{it}-\tfrac{1}{2}ite^{it}-\tfrac{1}{4}e^{-it}$$

Multiplying throughout by the integrating factor e^{it},

$$\frac{dr_4}{dt}e^{it}+ir_4(t)e^{it}=\frac{1}{4}e^{2it}-\frac{1}{2}ite^{2it}-\frac{1}{4}\Longrightarrow\frac{d(r_4(t)e^{it})}{dt}=\frac{1}{4}e^{2it}-\frac{1}{2}ite^{2it}-\frac{1}{4}$$

$$\Longrightarrow r_4(t)e^{it}=-\frac{1}{4}ie^{2it}-\frac{t}{4}e^{2it}-\frac{t}{4}+c$$

$$\Longrightarrow r_4(t)=-\tfrac{1}{4}ie^{it}-\tfrac{1}{4}te^{it}-\tfrac{1}{4}te^{-it}+\tfrac{1}{4}ie^{-it}\ ,$$

$$e^{At}=p_0r_1+p_1r_2+p_2r_3+p_3r_4$$

$$e^{At}=\begin{pmatrix} \cos(t) & \sin(t) & 0 & 0 \\ -\sin(t) & \cos(t) & 0 & 0 \\ \frac{t}{2}\sin(t) & \frac{1}{2}\sin(t)-\frac{t}{2}\cos(t) & \cos(t) & -\sin(t) \\ \frac{1}{2}\sin(t)+\frac{t}{2}\cos(t) & \frac{t}{2}\sin(t) & -\sin(t) & \cos(t) \end{pmatrix}$$

3. Non-homogeneous linear ordinary differential equations

Theorem 3.1: If $\emptyset(t)$ is a fundamental matrix for $\dot{x}(t)=A(t)x(t)$, then the solution of

$$\begin{cases} \dot{x}(t) = A(t)x(t) + b(t) \\ x(t) = \xi \end{cases}$$

is

$$x(t) = \emptyset(t)(\emptyset^{-1}(\lambda)\xi + \int_{\lambda}^{t} \emptyset^{-1}(s)b(s)ds)$$

Proof:

Put $x(t) = \emptyset(t)y(t)$, so that

$$\dot{x}(t) = \emptyset(t)\dot{y}(t) + \dot{\emptyset}(t)y(t)$$
$$= \emptyset(t)\dot{y}(t) + \big(A(t)\emptyset(t)\big)y(t).$$

Because $\emptyset(t)$ is a fundamental matrix also $\dot{x}(t) = A(t)x(t) + b = A(t)\big(\emptyset(t)y(t)\big) + b$ so that $\emptyset(t)\dot{y}(t) = b, \dot{y}(t) = \emptyset^{-1}(t)b(t)$, with $\xi = \emptyset(\lambda)y(\lambda)$ because $x(\lambda) = \xi$ so $y(\lambda) = \emptyset^{-1}(\lambda)\xi$, $y(t) = y(\lambda) + \int_{\lambda}^{t} \dot{y}(s)ds = \emptyset^{-1}(\lambda)\xi + \int_{\lambda}^{t} \dot{y}(s)ds$ so $x(t) = \emptyset(t)y(t) = \emptyset(t)\left(\emptyset^{-1}(\lambda)\xi + \int_{\lambda}^{t} \dot{y}(s)ds\right) = \emptyset(t)(\emptyset^{-1}(\lambda)\xi + \int_{\lambda}^{t} \emptyset^{-1}(s)b(s)ds)$.

Example 3.1: Find the unique solution of $\begin{cases} \dot{x}(t) = Ax(t) + b(t) \\ x(0) = \xi. \end{cases}$ such that

$$A = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}, b(t) = \begin{bmatrix} e^{5t} \\ 4 \end{bmatrix}, \xi = \begin{bmatrix} 9 \\ 4 \end{bmatrix}.$$

Solution: $e^{At} = \sum_{k=1}^{2} r_k(t)P_{k-1} = r_1(t)P_0 + r_2(t)P_1$

$$P_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and}$$

$P_1 = (A - \lambda_1)P_0$, $P(\lambda) = \det(\lambda I - A) = \begin{vmatrix} 6-\lambda & -3 \\ 2 & 1-\lambda \end{vmatrix} = 0$. This implies that $\lambda_1 = 3$ and $\lambda_2 = 4$. $p_1 = \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix}$, $\frac{dr_1}{dt} = \lambda_1 r_1$, since $r_1(0) = 1$, $\frac{dr_1}{dt} = 3r_1$ which implies $r_1(t) = e^{3t}$ and $\frac{dr_2}{dt} = \lambda_2 r_2 + r_1$ since $r_2(0) = 0$, $\frac{dr_2}{dt} - 4r_2 = e^{3t}$ which implies $r_2(t) = (-e^{3t} + e^{4t})$.

Thus,

$$e^{At} = \begin{bmatrix} -2e^{3t} + 3e^{4t} & 3e^{3t} - 3e^{4t} \\ -2e^{3t} + 2e^{4t} & 3e^{3t} - 2e^{4t} \end{bmatrix}.$$

$\emptyset$ is a fundamental matrix of

$$\dot{x}(t) = A(t)x(t) + b(t).$$

$x(t) = \emptyset(t)(\emptyset^{-1}(\lambda)\xi + \int_{\lambda}^{t} \emptyset^{-1}(s)b(s)ds)$. Let $x_1(t) = \emptyset(t)(\emptyset^{-1}(\lambda)\xi$

and

$$x_2(t) = \emptyset(t)\int_{\lambda}^{t} \emptyset^{-1}(s)b(s)ds.$$

$$\emptyset^{-1}(x) = \frac{1}{e^{7t}}\begin{bmatrix} 3e^{3t} - 2e^{4t} & -(3e^{3t} - 3e^{4t}) \\ 2e^{3t} - 2e^{4t} & -(2e^{3t} - 3e^{4t}) \end{bmatrix}$$

$$\emptyset^{-1}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

Thus,

$$x_1(t) = \begin{bmatrix} -2e^{3t} + 3e^{4t} & 3e^{3t} - 3e^{4t} \\ -2e^{3t} + 2e^{4t} & 3e^{3t} - 2e^{4t} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} -6e^{3t} + 15e^{4t} \\ -6e^{3t} + 10e^{4t} \end{bmatrix},$$

$$x_2(t) =$$

$$\begin{bmatrix} -2e^{3t} + 3e^{4t} & 3e^{3t} - 3e^{4t} \\ -2e^{3t} + 2e^{4t} & 3e^{3t} - 2e^{4t} \end{bmatrix}\int_0^t \frac{1}{e^{7s}}\begin{bmatrix} 3e^{3s} - 2e^{4s} & -(3e^{3s} - 3e^{4s}) \\ 2e^{3s} - 2e^{4s} & -(2e^{3s} - 3e^{4s}) \end{bmatrix}\begin{bmatrix} e^{5s} \\ 4 \end{bmatrix}ds$$

$$= \begin{bmatrix} (3e^{3t} - 3e^{4t})(2e^{t} - e^{2t} - 4e^{-3t} + 2e^{4t} + 1) - (2e^{3t} - 3e^{4t})(3e^{t} - e^{2t} - 4e^{-3t} + 4e^{4t} + 5) \\ (3e^{3t} - 2e^{4t})(2e^{t} - e^{2t} - 4e^{-3t} + 2e^{4t} + 1) - (2e^{3t} - 2e^{4t})(3e^{t} - e^{2t} - 4e^{-3t} + 3e^{4t} + 5) \end{bmatrix}$$

Then $x(t) = x_1(t) + x_2(t) =$

$$\begin{bmatrix} (-6e^{3t} + 15e^{4t}) + (3e^{3t} - 3e^{4t})(2e^{t} - e^{2t} - 4e^{-3t} + 2e^{4t} + 1) - (2e^{3t} - 3e^{4t})(3e^{t} - e^{2t} - 4e^{-3t} + 3e^{4t} + 5) \\ (-6e^{3t} + 10e^{4t}) + (3e^{3t} - 2e^{4t})(2e^{t} - e^{2t} - 4e^{-3t} + 2e^{4t} + 1) - (2e^{3t} - 2e^{4t})(3e^{t} - e^{2t} - 4e^{-3t} + 3e^{4t} + 5) \end{bmatrix}$$

Example 3.2. Find the unique solution of $\dot{x} = Ax(t) + b(t)$ such

$$A = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e^{5t} \\ 4 \end{pmatrix}$$

Solution:

$$det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -3 \\ 2 & 1-\lambda \end{vmatrix} = 0 \textit{ implies } \lambda_1 = 3, \lambda_2 = 4$$

$$x_c = a_1(t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t} + a_2(t)\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{4t}$$

Where a_1 and a_2 are any arbitrary constants. Let x_p be in the from

$$x_p = a_1(t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t} + a_2(t)\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{4t} \qquad (3.1)$$

Substitute in (3.1) , we get

$$3a_1(t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t} + \dot{a}_1(t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t} + 4a_2(t)\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{4t} + \dot{a}_2(t)\begin{pmatrix} 3 \\ 2 \end{pmatrix}e^{3t}$$

$$= \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a_1(t)e^{3t} + 3a_2(t)e^{4t} \\ 3a_1(t)e^{3t} + 2a_2(t)e^{4t} \end{pmatrix} + \begin{pmatrix} e^{5t} \\ 4 \end{pmatrix}$$

$$\dot{a}_1(t)\begin{pmatrix}1\\1\end{pmatrix}e^{3t} + \dot{a}_2(t)\begin{pmatrix}3\\2\end{pmatrix}e^{3t} + \begin{pmatrix}3a_1(t)e^{3t} + 12a_2(t)e^{4t}\\3a_1(t)e^{3t} + 8a_2(t)e^{4t}\end{pmatrix} =$$

$$= \begin{pmatrix}3a_1(t)e^{3t} + 12a_2(t)e^{4t}\\3a_1(t)e^{3t} + 8a_2(t)e^{4t}\end{pmatrix} + \begin{pmatrix}e^{5t}\\4\end{pmatrix}$$

$$\dot{a}_1(t)e^{3t} + 3\dot{a}_2(t)e^{4t} = e^{5t}$$
$$\dot{a}_1(t)e^{3t} + 2\dot{a}_2(t)e^{4t} = 4$$

Solve these equations for $\dot{a}_1(t)$, $\dot{a}_2(t)$, we get,

$\dot{a}_2(t) = (e^{5t} - 4)e^{-4t}$

$\dot{a}_1(t) = (-2e^{5t} + 12)e^{-4t}$ hence, $a_2(t) = \int(e^t - 4e^{-4t})\,dt =$

$a_2(t) = e^t + e^{-4t}$, $a_1(t) = \int(-2e^{2t} - 4e^{-3t})dt = -e^{2t} - 4e^{-3t}$,

$$x_p = (-e^{2t} - 4e^{-3t})\begin{pmatrix}1\\1\end{pmatrix}e^{3t} + (e^t + e^{-4t})\begin{pmatrix}3\\2\end{pmatrix}e^{4t}$$

$$x_p = (-e^{5t} - 4)\begin{pmatrix}1\\1\end{pmatrix} + (e^{5t} + 1)\begin{pmatrix}3\\2\end{pmatrix}$$

The general solution is

$$\begin{pmatrix}x\\y\end{pmatrix} = a_1(t)\begin{pmatrix}1\\1\end{pmatrix}e^{3t} + a_2(t)\begin{pmatrix}3\\2\end{pmatrix} + e^{4t}(-e^{5t} - 4)\begin{pmatrix}1\\1\end{pmatrix} + (e^{5t} + 1)\begin{pmatrix}3\\2\end{pmatrix}$$

$$x = a_1(t)e^{3t} + 3a_2(t)e^{4t} + 2e^{5t} - 1$$
$$y = a_1(t)e^{3t} + 2a_2(t)e^{4t} + e^{5t} - 2$$

4. Non-linear ordinary differential equations and some Applications

If we have a system of non-linear differential equations (4.1) is called Hopf oscillators as follow:

$$\left.\begin{aligned}\dot{x} &= a(\mu - r^2)x - \omega y\\ \dot{y} &= b(\mu - r^2)y + \omega x\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (4.1)$$

Where $r = \sqrt{x^2 + y^2}$, $\mu, \omega, a, b > 0$. μ and ω are the amplitude and the frequency respectively. Instead of a and b, the equation (4.1) can be written as

$$\left.\begin{aligned}\dot{x} &= \gamma(\mu - r^2)x - \omega y\\ \dot{y} &= \gamma(\mu - r^2)y + \omega x\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.2)$$

We need to linearize the function by Jacobean matrix.

$$\left.\begin{aligned}\dot{x} &= \gamma(\mu - r^2)x - \omega y\\ \dot{y} &= \gamma(\mu - r^2)y + \omega x\end{aligned}\right\}$$

A close look into the stabilities and bifurcation both in Cartesian coordinates and phase space reveals that the origin point is the fixed point. By linearizing this system at the origin point, we obtain

$$J(0,0) = \begin{bmatrix}\gamma\mu & -\omega\\ \omega & \gamma\mu\end{bmatrix}$$

Then, we can obtain the following eigenvalues

$$(\lambda - \gamma\mu)^2 + \omega^2 = 0 \Rightarrow \lambda_{1,2} = \gamma\mu \pm i\omega$$
$$\lambda_1 = \gamma\mu + i\omega$$
$$\lambda_2 = \gamma\mu - i\omega$$

where $\gamma > 0$. Interestingly, If $\mu < 0$, then our system is said to be asymptotically stable. If $\mu = 0$, on the other hand, then these solutions are said to be periodic solutions and limit cycles. However, when $\mu > 0$, this system will be unstable. This system, in phase space, is given by

$$\dot{r} = \gamma(\mu - r^2)r$$
$$\dot{\emptyset} = \omega$$

From the phase space standpoint, we can see that if $\gamma = 1$, we obtain

$$\dot{r} = (\mu - r^2)r \;\&\; \dot{\emptyset} = \omega$$
$$r \geq 0$$

The only critical point of this system is that $r^* = 0$, which is the original point. the trajectories depend on the value of ω; if $\omega > 0$, then, the trajectories will move clockwise around the origin, otherwise the trajectories will move anticlockwise around the origin. If $\mu = 0$ and $\omega > 0$, then $\dot{r} = -r^3$. For nonzero r , we have $\dot{r} < 0$, hence there are no closed orbits and all trajectories approach the origin as $t \to \infty$ as seen in figure. 1.

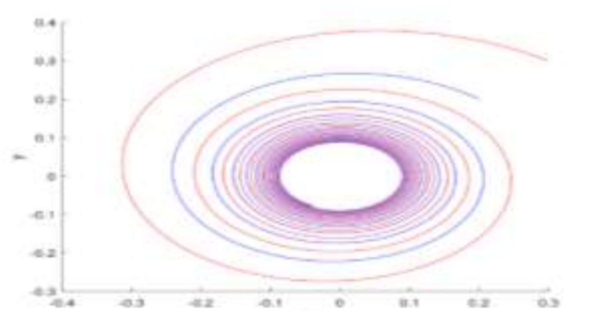

Fig. 1. Clockwise Trajectory Movement: Red curve correspond to the initial condition ($x_1(0)$, $y_1(0)$)=(0.3, 0.3), and the blue solution corresponds to the initial condition ($x_1(0)$, $y_1(0)$)=(0.2, 0.2). These different solutions correspond to the values $\mu = 0$ and $\omega = \gamma = 1$.

5. Conclusion and Future Directions

In this paper, we display a comparison between Via-Leonard II theorem and Putzer algorithm theorem for solving some homogeneous linear ordinary differential equations and nonlinear ordinary differential equations. The results obtained show that gives the unique solution. On the other hand, the calculations are simpler and faster than in traditional techniques. The Via-Leonard II theorem and Putzer algorithm theorem are two different mathematical concepts. The Via-Leonard II theorem is a result in numerical analysis that provides error bounds for approximating solutions of ordinary differential equations (ODEs) using a collocation method.

Finally, in this paper, the obtained that given the exact solution "same solution" on comparison between Via-Leonard II theorem and Putzer algorithm theorem for solving some homogeneous linear ordinary differential equations and nonlinear ordinary differential equations. Future part will be concerned deeper analysis of astudy the comparison between the Leonard II theorem and Putzer algorithm for solving some linear and non-linear difference equations.

6. References

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