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A New Application of Sawi Transform for Solving Volterra Integral Equations and Volterra Integro-differential Equations

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Abstract

In this paper, a new application of Sawi transform is given for solving Volterra integral equations and Volterra integro-differential equations. In the application section of this paper, some applications are given to explain the importance of Sawi transform for solving the indicated equations. The results show that Sawi transform is a very useful integral transform for solving these equations.

Keywords: Volterra Integral Equations; Volterra Integro-differential Equations; Sawi Transform.

المستخلص

في هذه الورقة قدمنا تطبيقاً جديداً لتحويل ساوي لحل معادلات فولتيرا التكاملية ومعادلات فولتيرا التكاملية–التفاضلية وفي بند التطبيقات في هذه الورقة أعطيت بعض التطبيقات لتشرح أهمية تحويل ساوي لحل المعادلات المشار إليها. وقد وضحت النتائج أن تحويل ساوي تحويل تكاملي مفيد جداً لحل المعادلات المذكورة.

Introduction

In modern times, integral transforms, Laplace transform [12, 16,20,21,24,29,30,31,32], Fourier transform [24], Hankel transform [24], Mellin transform [24], Mahgoub transform [7,11,13,23,25], Kamal transform [1,2,5,8,10,22], Elzaki transform [6,15,17,18,19,30], Aboodh transform [3,4,14], Mohand transform [4,27],Sawi transform [26], Sadik transform [28] and Hermite transform [24] etc. have very useful role in mathematics, physics, chemistry, social science, biology, astronomy, nuclear science, electrical and mechanical engineering for solving the advanced problems of these fields. Many scholars use these transforms to solve problems of differential equations, partial differential equations, integral equations, integro-differential equations, partial integro-differential equations, delay differential equations and population growth and decay problems, and the mechanics and

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electrical circuit problems. In this paper, we concentrate mainly on an application of the Sawi transform to solve some linear Volterra integral equations and linear Volterra integro-differential equations.

Definitions and Standard Results

Sawi Transform

Definition: A new transform called the Sawi transform defined for function of exponential order, we consider functions in the set A defined by:

$$A=\left\{ f\left(t\right):\exists\ M,K_1,K_2>0,\left|f\left(t\right)\right|<M\,e^{\frac{|t|}{k_j}},if\ \ t\in(-1)^j\times[0,\infty)\right\} \quad (1)$$

For a given function in the set A, the constant M must be a finite number. K_1,K_2 may be finite or infinite. Sawi Transform is denoted by the operator $S\left(\cdot\right)$; defined by the integral equations

$$S\left[f\left(t\right)\right]=R\left(v\right)=\frac{1}{v^2}\int_0^\infty f\left(t\right)e^{\frac{-t}{v}}\,dt\quad,\ t\geq 0,\ K_1\leq v\leq K_2 \qquad (2)$$

Some Properties of Sawi Transform

Linearity Property of Sawi Transform

If $S\left[f_1(t)\right]=R_1(v)$ *and* $S\left[f_2(t)\right]=R_2(v)$ then

$$S\left[a\,f_1(t)+b\,f_2(t)\right]=aS\left[f_1(t)\right]+bS\left[f_2(t)\right]=aR_1(v)+b\,R_2(v) \quad (3)$$

Where a, b are arbitrary constants.

Sawi Transform of the Derivative of the Function $f\left(t\right)$

Theorem: let $S\left[f\left(t\right)\right]=R\left(v\right)$ then

$$(i)\quad S\left[f'(t)\right]=\frac{1}{v}R\left(v\right)-\frac{1}{v^2}f\left(0\right) \qquad (4)$$

$$(ii)\quad S\left[f''(t)\right]=\frac{1}{v^2}R(v)-\frac{1}{v^2}f'(0)-\frac{1}{v^3}f(0) \tag{5}$$

$$(iii)\quad S\left[f^{(n)}(t)\right]=\frac{1}{v^n}R(v)-\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{v^{n-k+1}} \tag{6}$$

Sawi Transform of Integral of a Function $f(t)$

Theorem: let $S\left[f(t)\right]=R(v)$ then

$$S\left[\int_0^t f(t)dt\right]=v\,R(v) \tag{7}$$

Proof: let $H(t)=\int_0^t f(t)dt \quad\Rightarrow H'(t)=f(t),\quad H(0)=0$

Now using Sawi transform of the derivative of a function, we get

$$S\left[H'(t)\right]=\frac{1}{v}S\left[H(t)\right]-\frac{1}{v^2}H(0)\Rightarrow\quad S\left[H'(t)\right]=\frac{S\left[H(t)\right]}{v} \tag{8}$$

$$\Rightarrow\quad S\left[H(t)\right]=v\,S\left[H'(t)\right]=v\,S\left[f(t)\right]=v\,R(v) \tag{9}$$

$$\Rightarrow\quad S\left[\int_0^t f(t)dt\right]=v\,R(v) \tag{10}$$

Change of Scale Property

Theorem: let $S\left[f(t)\right]=R(v)$ then $S\left[f(at)\right]=a\,R(av)$

Proof:

$$\because S\left[f(t)\right]=\frac{1}{v^2}\int_0^\infty f(t)\,e^{\frac{-t}{v}}\,dt\ \Rightarrow S\left[f(at)\right]=\frac{1}{v^2}\int_0^\infty f(at)\,e^{\frac{-t}{v}}\,dt \tag{11}$$

Let $at=u\quad\Rightarrow\quad t=\frac{u}{a}\Rightarrow\quad dt=\frac{1}{a}\,du$

$$\Rightarrow \quad S\left[f\left(u\right)\right]=\frac{1}{v^2}\int_0^\infty f\left(u\right)e^{\frac{-u}{av}}\ \ \frac{1}{a}du=\frac{1}{a}\frac{1}{v^2}\int_0^\infty f\left(u\right)e^{\frac{-u}{(av)}}\ \ du \qquad (12)$$

$$\Rightarrow \quad S\left[f\left(u\right)\right]=a\ \frac{1}{\left(av\right)^2}\int_0^\infty f\left(u\right)e^{\frac{-u}{(av)}}\ \ du=aR\left(av\right) \qquad (13)$$

Convolution Property

Theorem: let $f\left(t\right),\ g\left(t\right)\in A\ \ and\ \ \ S\left[f\left(t\right)\right]=F\left(v\right)\ \ ,\ S\left[g\left(t\right)\right]=G\left(v\right)$ then

$$S\left[f\left(t\right)*g\left(t\right)\right]=v^2F\left(v\right)G\left(v\right) \qquad (14)$$

where $f\left(t\right)*g\left(t\right)=\int_0^t f\left(t-x\right)g\left(x\right)dx=\int_0^t f\left(x\right)g\left(t-x\right)dx$

Proof: Consider $F\left(v\right)=\frac{1}{v^2}\int_0^\infty e^{\frac{-r}{v}}f\left(r\right)\ dr\ \ ,G\left(v\right)=\frac{1}{v^2}\int_0^\infty e^{\frac{-s}{v}}g\left(s\right)\ ds$

$$\Rightarrow F\left(v\right)G\left(v\right)=\frac{1}{v^4}\int_0^\infty e^{\frac{-r}{v}}f\left(r\right)\ dr\ \int_0^\infty e^{\frac{-s}{v}}g\left(s\right)\ \ ds=\frac{1}{v^4}\int_0^\infty\int_0^\infty e^{\frac{-1}{v}(r+s)}f\left(r\right)g\left(s\right)ds\,dr \qquad (15)$$

put $r+s=t\ \ \Rightarrow\ \ ds=dt$ and by changing the order of integration and solving we obtain

$$F\left(v\right)G\left(v\right)=\frac{1}{v^2}\left[\frac{1}{v^2}\int_0^\infty\int_0^t e^{\frac{-t}{v}}f\left(r\right)g\left(t-r\right)dr\,dt\right]=\frac{1}{v^2}S\left[f\left(t\right)*g\left(t\right)\right] \qquad (16)$$

$$\Rightarrow \quad S\left[f\left(t\right)*g\left(t\right)\right]=v^2F\left(v\right)G\left(v\right) \qquad (17)$$

Sawi Transform for Some Functions

Table 1. Sawi Transform functions.

$f\left(t\right)$	$S\left[f\left(t\right)\right]$
1	$\frac{1}{v}$
t	1
t^2	$2v$
t^n	$n!v^{n-1}$
e^{at}	$\frac{1}{v\left(1-av\right)}$

$\sin at$	$\dfrac{a}{1+a^2v^2}$
$\cos at$	$\dfrac{1}{v\left(1+a^2v^2\right)}$
$\sinh at$	$\dfrac{a}{1-a^2v^2}$
$\cosh at$	$\dfrac{1}{v\left(1-a^2v^2\right)}$

Applications

Example 1.

Consider linear Volterra integral equation of second kind

$$y(x)=1-x+\int_0^x (x-t)\,y(t)\,dt \tag{18}$$

Solution: Applying the Sawi transform to both sides we have

$$S\left[y(x)\right]=S[1]-S[x]+S\left\{\int_0^x (x-t)\,y(t)\,dt\right\} \tag{19}$$

$$R(v)=\frac{1}{v}-1+v^2R(v) \tag{20}$$

$$\Rightarrow \quad R(v)=\frac{1-v}{v\left(1-v^2\right)}=\frac{1}{v(1+v)} \tag{21}$$

Now using inverse Sawi transform, we get

$$y(x)=e^{-x} \tag{22}$$

Example 2.

Consider linear Volterra integral equation of second kind

$$y(x) = 3x^2 + \int_0^x \sin(x-t)\, y(t)\, dt \tag{23}$$

Solution: Applying the Sawi transform to both sides we have

$$S[y(x)] = 3S[x^2] + v^2 S[\sin x]\, S[y(x)] \tag{24}$$

$$R(v) = 6v + v^2 \frac{1}{1+v^2} R(v) \quad \Rightarrow R(v)\left[1 - \frac{v^2}{1+v^2}\right] = 6v \tag{25}$$

$$\Rightarrow R(v)\left[\frac{1}{1+v^2}\right] = 6v \quad \Rightarrow \quad R(v) = 6v + 6v^3 \tag{26}$$

$$\Rightarrow \; R(v) = 3[2v] + \frac{1}{4}\left[4!\, v^3\right] \tag{27}$$

Now using inverse Sawi transform, we get

$$y(x) = 3x^2 + \frac{1}{4} x^4 \tag{28}$$

Example 3.

Consider linear Volterra integral equation of second kind

$$y(x) = x + \frac{1}{6}\int_0^x (x-t)^3\, y(t)\, dt \tag{29}$$

Solution: Applying the Sawi transform to both sides we have

$$S[y(x)] = S[x] + \frac{1}{6} v^2 S[x^3] S[y(x)] \tag{30}$$

$$R(v) = 1 + v^4\, R(v) \tag{31}$$

$$\Rightarrow \; R(v) = \frac{1}{1-v^4} = \frac{1}{2}\frac{1}{1-v^2} + \frac{1}{2}\frac{1}{1+v^2} \tag{32}$$

Now using inverse Sawi transform, we get

$$y(x) = \frac{1}{2}[\sinh x + \sin x] \tag{33}$$

Example 4.

Consider linear Volterra integral equation

$$y(x) = 1 - \frac{1}{2}x^2 + \int_0^x y(t)\, dt \tag{34}$$

Solution: Applying the Sawi transform to both sides we have

$$R(v) = \frac{1}{v} - v + v\, R(v) \tag{35}$$

$$\Rightarrow \quad R(v) = \frac{1 - v^2}{v(1 - v)} = \frac{1}{v} + 1 \tag{36}$$

Now using inverse Sawi transform, we get

$$y(x) = 1 + x \tag{37}$$

Example 5.

Consider linear Volterra integral equation of first kind

$$\sin x = \int_0^x e^{(x-t)}\, y(t)\, dt \tag{38}$$

Solution: Applying the Sawi transform to both sides we have

$$\frac{1}{1 + v^2} = \frac{v^2}{v(1 - v)} R(v) \tag{39}$$

$$\Rightarrow \quad R(v) = \frac{1 - v}{v(1 + v^2)} = \frac{1}{v(1 + v^2)} - \frac{1}{1 + v^2} \tag{40}$$

Now using inverse Sawi transform, we get

$$y(x) = \cos x - \sin x \tag{41}$$

Example 6.

Consider linear Volterra integral equation of first kind

$$x^2 = \frac{1}{2}\int_0^x (x-t)\, y(t)\, dt \tag{42}$$

Solution: Applying the Sawi transform to both sides we have

$$2v = \frac{1}{2} v^2 R(v) \tag{43}$$

$$\Rightarrow \quad R(v) = \frac{4}{v} = 4\frac{1}{v} \quad \Rightarrow \qquad y(x) = 4 \tag{44}$$

Example 7.

Consider linear Volterra integral equation of first kind

$$x = \int_0^x e^{(x-t)}\, y(t)\, dt \tag{45}$$

Solution: Applying the Sawi transform to both sides we have

$$1 = \frac{v}{1-v} R(v) \;\Rightarrow\; R(v) = \frac{1}{v} - 1 \tag{46}$$

Now using inverse Sawi transform, we get

$$y(x) = 1 - x \tag{47}$$

Example 8.

Consider linear Volterra integro - differential equation of second kind

$$y'(x) = 2 + \int_0^x y(t)\; dt \quad with \quad y(0) = 2 \tag{48}$$

Solution: Applying the Sawi transform to both sides we have

$$\frac{R(v)}{v}-\frac{2}{v^2}=\frac{2}{v}+v\,R(v) \quad \Rightarrow R(v)\left(\frac{1}{v}-v\right)=\frac{2(v+1)}{v^2} \tag{49}$$

$$\Rightarrow R(v)=\frac{2(v+1)}{v\left(1-v^2\right)}=\frac{2}{v(1-v)} \tag{50}$$

Now Using inverse Sawi transform, we get

$$y(x)=2e^x \tag{51}$$

Example 9.

Consider linear Volterra integro-differential equation of second kind

$$y''(x)=-1-x+\int_0^x (x-t)\,y(t)\,dt \tag{52}$$

With $y(0)=1,\ y'(0)=1$

Solution: Applying the Sawi transform to both sides we have

$$\frac{R(v)}{v^2}-\frac{1}{v^2}-\frac{1}{v^3}=-\frac{1}{v}-1+v^2R(v) \tag{53}$$

$$\left[\frac{1-v^4}{v^2}\right]R(v)=\frac{v+1-v^2-v^3}{v^3}\Rightarrow R(v)=\frac{\left(1-v^2\right)+v\left(1-v^2\right)}{v\left(1-v^4\right)} \tag{54}$$

$$\Rightarrow R(v)=\frac{1}{v\left(1+v^2\right)}+\frac{1}{1+v^2} \tag{55}$$

Now using inverse Sawi transform, we get

$$y(x)=\cos x+\sin x \tag{56}$$

Example 10.

Consider linear Volterra integro-differential equation of second kind

$$y''(x) = \cosh x + \int_0^x e^{(x-t)}\, y(t)\, dt \qquad (57)$$

With $y(0)=1$, $y'(0)=1$

Solution: Applying the Sawi transform to both sides we have

$$\frac{R(v)}{v^2} - \frac{1}{v^2} - \frac{1}{v^3} = \frac{1}{v\left(1-v^2\right)} + \frac{v}{1+v} R(v) \qquad (58)$$

$$\left[\frac{1+v-v^3}{v^2(1+v)}\right] R(v) = \frac{v^2+v\left(1-v^2\right)+\left(1-v^2\right)}{v^3\left(1-v^2\right)} = \frac{1+v-v^3}{v^3(1+v)(1-v)} \qquad (59)$$

$$\Rightarrow\ R(v) = \frac{1}{v(1-v)} \qquad (60)$$

Now using Inverse Sawi transform, we get

$$y(x) = e^x \qquad (61)$$

Example 11.

Consider linear Volterra integro-differential equation of second kind

$$y'''(x) = -1 + \int_0^x y(t)\ dt \qquad (62)$$

With $y(0) = y'(0) = 1 \quad , \quad y''(t) = -1$

Solution: Applying the Sawi transform to both sides we have

$$\frac{R(v)}{v^3} - \frac{1}{v^4} - \frac{1}{v^3} + \frac{1}{v^2} = \frac{-1}{v} + v\ R(v) \qquad (63)$$

$$R(v)\left[\frac{1-v^4}{v^3}\right] = \frac{1+v-v^2-v^3}{v^4} = \frac{\left(1-v^2\right)+v\left(1-v^2\right)}{v^4} \qquad (64)$$

$$\Rightarrow \quad R(v) = \frac{\left(1-v^2\right)+v\left(1-v^2\right)}{v\left(1-v^2\right)\left(1+v^2\right)} = \frac{1}{v\left(1+v^2\right)} + \frac{1}{1+v^2} \qquad (65)$$

Now using inverse Sawi transform, we get

$$y(x) = \cos x + \sin x \qquad (66)$$

Conclusion

In this present work, the Sawi transform is used to solve Volterra integral equations and Volterra Integro-differential Equations. Furthermore, we present several properties and theorems of Sawi transform. To see the efficiency of Sawi transform, we applied this transform on eleven different examples. The results show that the Sawi transform method is an appropriate method for solving Volterra integral equations and Volterra Integro-differential Equations.

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