



## Metrizable Spaces with finitely many non-isolated points

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### ABSTRACT

A point  $x$  in a topological space  $X$  is said to be non-isolated if every open set containing  $x$  contains another point different from  $x$ . The aim of this paper is to give the topology and the metric induces that topology of a metrizable space  $X$  with exactly  $n$  non-isolated points .

**Keywords:** metrizable space; isolated point; free filter

### 1. Preliminaries

Let  $X$  be an infinite set, a space  $(X, \tau)$  is said to be metrizable if there exists a metric induces the topology  $\tau$  [1, 6]. A collection  $\mathfrak{F}$  of non- empty subsets of  $X$  is called a filter if  $F_1, F_2 \in \mathfrak{F}$  then  $F_1 \cap F_2 \in \mathfrak{F}$  and if  $G \in \mathfrak{F}$ ,  $G \subseteq X$  with  $F \subseteq G$ , then  $G \in \mathfrak{F}$ . A filter  $\mathfrak{F}$  in  $X$  is said to be free filter if  $\bigcap_{F \in \mathfrak{F}} F = \emptyset$ . A subcollection  $\mathcal{B}$  of a filter  $\mathfrak{F}$  is said to be a filter base for  $\mathfrak{F}$  if for any  $F \in \mathfrak{F}$  there exists  $C \in \mathcal{B}$  with  $C \subseteq F$  [1, 6]. A collection of subsets of a topological space  $X$  is said to be locally finite collection if every point in  $X$  has an open neighborhood which intersects only finitely many members of  $\mathcal{B}$ . A collection  $\mathcal{B}$  of subsets of a topological space  $X$  is said to be  $\sigma$  –

locally finite if  $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$  where  $\mathcal{B}_n$  is locally finite collection for all  $n$  [6]. A point  $x$  of a topological space  $X$  is called an isolated point if  $\{x\}$  is open. The following Theorems are respectively Theorem 2.2 and Theorem 2.1 of [5].

#### 1.1. Theorem

A space  $(X, \tau)$  is metrizable space with only one non- isolated point  $x_1$  iff

$\tau = p(X \setminus \{x_1\}) \cup \{\{x_1\}\} \cup \mathcal{F}$ , where  $\mathfrak{F}$  is a free filter in  $X \setminus \{x_1\}$  with countable filter base  $\{C_n\}_{n=1}^{\infty}$  with  $C_{n+1} \subsetneq C_n$  for all  $n$ ,  $\bigcap_{n=1}^{\infty} C_n = \emptyset$ , and  $P(X \setminus \{x_1\})$  is the power set of  $X \setminus \{x_1\}$ .

## 1.2. Theorem

$d: X \times X \rightarrow \mathbb{R}$  defined by

If  $X, \tau, x_1, \{C_n\}_{n=1}^\infty$  and  $\mathfrak{F}$  are as in the above

Theorem, then the function

$$d(x, y) = \begin{cases} 1 & \text{if } x, y \in X \setminus \{x_1\}, x \neq y, x \notin C_1 \text{ or } y \notin C_1 \\ 1 & \text{if } (x = x_1, y \notin C_1) \text{ or } (y = x_1, x \notin C_1) \\ \frac{1}{n+1} & \text{if } (x = x_1, y \in C_1) \text{ or } (y = x_1, x \in C_1) \\ & \text{where } n \text{ is the least integer such that } x \notin C_n \text{ (resp. } y \notin C_n) \\ \max\left\{\frac{1}{n+1}, \frac{1}{m+1}\right\} & \text{if } x \neq y; x, y \in C_1 \\ & \text{and } m, n \text{ are respectively the least} \\ & \text{integers such that } x \notin C_m, y \notin C_n \\ 0 & \text{if } x = y, x, y \in X \end{cases}$$

is a metric and induces the topology  $\tau$ .

## 1.3. Remark

Since the metric defined above is depending on the collection  $\{C_n\}_{n=1}^\infty$  and the point  $x_1$  we call it the metric associated with the collection  $\{C_n\}_{n=1}^\infty$  and the point  $x_1$ .

The aim of this paper is to give the topology and a metric induces that topology for any metrizable space with finitely many of non-isolated points.

## 2. The Main results

First, we start with metrizable spaces with only two non-isolated points.

### 2.1. Theorem

if  $x_1, x_2 \in X, x_1 \neq x_2$

$\tau = P(X \setminus \{x_1, x_2\}) \cup \{\{x_1\} \cup F : F \in \mathfrak{F}_1\} \cup \{\{x_2\} \cup G : G \in \mathfrak{F}_2\}$

$\{\{x_1, x_2\} \cup F : F \in \mathfrak{F}_1\} \cup \{\{x_1, x_2\} \cup G : G \in \mathfrak{F}_2\}$ .

Where  $\mathfrak{F}_1, \mathfrak{F}_2$  are free filters in  $X \setminus \{x_1, x_2\}$  with filter base  $\{C_{1n}\}_{n=1}^\infty$ ,

$\{C_{2n}\}_{n=1}^\infty$  Respectively such that  $C_{1n+1} \subsetneq C_{1n}, C_{2n+1} \subsetneq C_{2n}$  for all  $n$ ,

$\bigcap_{n=1}^\infty C_{1n} = \bigcap_{n=1}^\infty C_{2n} = \varnothing$  and  $C_{11} \cap C_{21} = \varnothing$ , then  $\tau$  is a topology on  $X$

and  $(X, \tau)$  is a  $T_4$  space.

### Proof

It is routine to check that  $\tau$  is a topology on  $X$ . To show that  $(X, \tau)$  is

Hausdorff, let  $x, y \in X$  with  $x \neq y$ . If  $x = x_1, y \in X \setminus \{x_1, x_2\}$

then since  $\mathfrak{F}_1$  is a free filter so there exists  $F \in \mathfrak{F}_1$  such that  $y \notin F$  and so  $\{x_1\} \cup F, \{y\}$  are disjoint open sets containing  $x_1$  and  $y$  respectively. Similarly if  $x = x_2, y \in X \setminus \{x_1, x_2\}$ . if  $x = x_1, y = x_2$ , then  $\{x_1\} \cup C_{11}, \{x_2\} \cup C_{21}$  are disjoint open sets containing  $x_1$  and  $x_2$  respectively. So  $(X, \tau)$  is Hausdorff. To show  $(X, \tau)$  is normal space let  $A, B$  be any two disjoint closed sets, then we have the following cases :

- If  $A, B$  are subsets of  $X \setminus \{x_1, x_2\}$ , then  $A, B$  are both open.
- If  $x_1 \in A, x_2 \notin B$  then  $B$  is an open set and so  $B$  and  $B^c$  are two disjoint open sets containing  $B$  and  $A$  respectively. Similarly if  $x_2 \in B, x_1 \notin A$
- If  $x_1 \in A, x_2 \in B$  then  $A^c$  is an open set.  $B \subset A^c$  So for any  $b \in B$  there exists with  $F_b \in \mathfrak{F}$ . with  $\{b\} \cup F_b \subset A^c$ . Let  $K = \bigcup_{b \in B} \{b\} \cup F_b$ , then  $K$  is a clopen (closed and open) set containing  $B$  and disjoint from  $A$ . Therefore  $(X, \tau)$  is a normal space and hence  $(X, \tau)$  is a  $T_4$  space.

The following Theorem is Theorem 23.9 of [6] .

## 2.2. Theorem

A topological space  $X$  is metrizable if and only if it is  $T_3$  and has a  $\sigma$  – locally finite base.

## 2.3. Theorem

If  $\tau$  as in Theorem 2.1, then  $(X, \tau)$  is a metrizable space .

### Proof:

Let  $\{C_{1n}\}_{n=1}^{\infty}, \{C_{2n}\}_{n=1}^{\infty}$  be as in Theorem 2.1 and let

$$\begin{aligned}\mathfrak{B}_1 &= \{\{x\} : x \in X \setminus \{x_1, x_2\}, x \notin C_{11} \cup C_{21}\} \cup \{\{x_1\} \cup C_{11}\} \cup \{\{x_2\} \cup C_{21}\} \\ &\dots \\ \mathfrak{B}_n &= \{\{x\} : x \in X \setminus \{x_1, x_2\}, x \notin C_{1n} \cup C_{2n}\} \\ &\cup \{\{x_1\} \cup C_{1n}\} \cup \{\{x_2\} \cup C_{2n}\}.\end{aligned}$$

Then  $\mathfrak{B}_n$  is locally finite collection for all  $n$  and  $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$  is a  $\sigma$  – locally finite collection. Since  $\mathfrak{B}$  contains the collection

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, x, y \in X \setminus \{x_1, x_2\}, (x, y) \notin C_{11} \times C_{11} \text{ and } (x, y) \notin C_{21} \times C_{21} \\ 0 & \text{if } x = y \\ 1 & \text{if } (x = x_1, y \notin C_{11}) \text{ or } (y = x_1, x \notin C_{11}) \\ 1 & \text{if } (x = x_2, y \notin C_{21}) \text{ or } (y = x_2, x \notin C_{21}) \\ \frac{1}{n+1} & \text{if } x = x_1, y \in C_{11} \text{ and } n \text{ is the least integer such that } y \notin C_{1n} \\ & \text{or} \\ & y = x_1, x \in C_{11} \text{ and } n \text{ is the least integer such that } x \notin C_{1n} \\ \frac{1}{m+1} & \text{if } x = x_2, y \in C_{21} \text{ and } m \text{ is the least integer such that } y \notin C_{2m} \\ & \text{or} \\ & y = x_2, x \in C_{21} \text{ and } m \text{ is the least integer such that } x \notin C_{2m} \\ \max\left\{\frac{1}{n+1}, \frac{1}{m+1}\right\} & \text{if } x, y \in C_{11} \text{ and } n, m \text{ are respectively the least integers} \\ & \text{such that } x \notin C_{1n}, y \notin C_{1m} \\ \max\left\{\frac{1}{n+1}, \frac{1}{m+1}\right\} & \text{if } x, y \in C_{21} \text{ and } n, m \text{ are respectively the least integers} \\ & \text{such that } x \notin C_{2n}, y \notin C_{2m} \end{cases}$$

Then  $d$  is a metric and induces the topology  $\tau$ .

### Proof

Let  $d_1$  be the metric corresponding to the collection  $\{C_{1n}\}_{n=1}^{\infty}$  and the point

$\{x\} : x \in X \setminus \{x_1, x_2\} \cup \{\{x_1\} \cup C_{1n}\}_{n=1}^{\infty} \cup \{\{x_2\} \cup C_{2n}\}_{n=1}^{\infty}$ . So  $\mathfrak{B}$  is a base for the topology and since  $(X, \tau)$  is  $T_4$ , so by the above Theorem  $(X, \tau)$  is a metrizable space .

## 2.2. Lemma

If  $d_1, d_2$  are two metrics on  $X$  and  $A, B$  are two non-empty subsets of  $X$  such that

$X = A \cup B, A \cap B = \emptyset$  and  $d_1(x, y) = d_2(x, y)$  if  $x \in A$  or  $y \in A$  then the function  $d: X \times X \rightarrow \mathbb{R}$  defined by

$$d(x, y) = \begin{cases} d_1(x, y) & \text{if } x \in A \text{ or } y \in A \\ d_2(x, y) & \text{if } x \in B, y \in B \end{cases}$$

is a metric on  $X$ .

### Proof

The proof follows since

$X \times X = (A \times A) \cup (B \times A) \cup (A \times B) \cup (B \times B)$  and  $d_1, d_2$  are two metrics on  $X$ .

## 2.3 Theorem

If  $\tau, \{C_{1n}\}_{n=1}^{\infty}$  and  $\{C_{2n}\}_{n=1}^{\infty}$  as in Theorem 2.1 and  $X \times X \rightarrow \mathbb{R}$  defined by

$d_1$  and  $d_2$  be the metric corresponding to the collection  $\{C_{2n}\}_{n=1}^{\infty}$  and the point  $x_2$ ,  
(See Remark 1.3).

Let  $A = X \setminus (C_{21} \cup \{x_2\})$ ,  $B = C_{21} \cup \{x_2\}$ , since  $A, B$  are two disjoint non empty subsets with  $X = A \cup B$ ,  $d_1(x, y) = d_2(x, y)$  if  $x \in A$  or  $y \in A$  and

$$d(x, y) = \begin{cases} d_1(x, y) & \text{if } x \in A \text{ or } y \in A \\ d_2(x, y) & \text{if } x \in B, y \in B \end{cases}$$

So by Lemma 2.2  $d$  is a metric on  $X$  and it is routine to show that  $d$  induces the topology  $\tau$ .

#### 2.4. Remarks:

i) The above metric can be written in the following way

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, x, y \in X \setminus \{x_1, x_2\}, (x, y) \notin C_{i1} \times C_{i1} \text{ for } i = 1, 2 \\ 0 & \text{if } x = y \\ 1 & \text{if } (x = x_i, y \notin C_{i1}) \text{ or } (y = x_i, x \notin C_{i1}) \text{ for } i = 1, 2 \\ \frac{1}{n+1} & \text{if } x = x_i, y \in C_{i1} \text{ and } n \text{ is the least integer such that } y \notin C_{in} \\ & \text{or} \\ & y = x_i, x \in C_{i1} \text{ and } n \text{ is the least integer such that } x \notin C_{in} \text{ for } i = 1, 2 \\ \max\left\{\frac{1}{n+1}, \frac{1}{m+1}\right\} & \text{if } x, y \in C_{i1} \text{ and } n, m \text{ are respectively the least integers} \\ & \text{such that } x \notin C_{in}, y \notin C_{im} \text{ for } i = 1, 2 \end{cases}$$

ii) For any  $x \in X$ ,  $\varepsilon > 0$  we have

$$B_\varepsilon(x) = \begin{cases} \{x\} & \text{if } x \notin C_{11} \cup C_{21}, x \neq x_1, x \neq x_2, \varepsilon = 1 \\ \{x\} & \text{if } x \in C_{11}, \varepsilon = \frac{1}{n+2} \text{ where } n \text{ is the least integer such that } x \in C_{1n} \\ \{x\} & \text{if } x \in C_{21}, \varepsilon = \frac{1}{m+2} \text{ where } m \text{ is the least integer such that } x \in C_{1m} \\ \{x_1\} \cup C_{1n} & \text{if } x = x_1, \varepsilon = \frac{1}{n} \\ \{x_2\} \cup C_{2m} & \text{if } x = x_2, \varepsilon = \frac{1}{m} \end{cases}$$

so as base for the metric topology we have the collection

$$\{\{x\} : x \neq x_1, x \neq x_2\} \cup \{\{x_1\} \cup C_{1n}\}_{n=1}^\infty \cup \{\{x_2\} \cup C_{2n}\}_{n=1}^\infty$$

#### 2.5. Notation

Let  $\Omega$  denotes the collection of all non-empty subsets of the set  $\{x_1, x_2, \dots, x_n\}$ .

The proof of the following Theorem is similar to the proof of Theorem 2.1

#### 2.6. Theorem

If  $x_1, x_2, \dots, x_n$  are elements of  $X$ ,

$$\tau = p(X \setminus \{x_1, x_2, \dots, x_n\}) \cup [\bigcup_{i=1}^n [\bigcup_{A \in \Omega} \{A \cup F : F \in \mathfrak{F}_i\}]]$$

where  $\mathfrak{F}_i$  is a free filter in

$X \setminus \{x_1, x_2, \dots, x_n\}$  with countable filter base, then  $\tau$

is a topology on  $X$  and is  $(X, \tau)$  a  $T_4$  space.

#### 2.7. Lemma

If  $x_1, x_2, \dots, x_n$  are distinct points in a metric space  $X$ , then there exists  $\varepsilon > 0$  such that  $B_\varepsilon(x_i) \cap B_\varepsilon(x_j) = \emptyset$  for  $i \neq j$ .

#### 2.8. Theorem

A space  $(X, \tau)$  with exactly  $n$  non-isolated points  $x_1, x_2, \dots, x_n$  is metrizable iff

$$\tau = p(X \setminus \{x_1, x_2, \dots, x_n\}) \cup [\bigcup_{i=1}^n [\bigcup_{A \in \Omega} \{A \cup F : F \in \mathfrak{F}_i\}]]$$

where  $\mathfrak{F}_i$  is a free filter in  $X \setminus \{x_1, x_2, \dots, x_n\}$  with countable filter base.

### Proof

$\Rightarrow$  Let  $X$  be a metrizable space with  $x_1, x_2, \dots, x_n$  as the only non-isolated points. Using Lemma 2.7 we

choose  $\varepsilon > 0$  such that  $B_\varepsilon(x_i) \cap B_\varepsilon(x_j) = \emptyset$  for  $i \neq j$

and let  $C_{ik} = B_{\varepsilon/k}(x_i) \setminus \{x_1, x_2, \dots, x_n\}$  for all  $k=1, 2, \dots, n$  then  $C_{ik} \cap X \setminus \{x_1, x_2, \dots, x_n\} = \emptyset$ ,

$$C_{ik+1} \subsetneq C_{ik} \text{ for all } k \text{ and } \bigcap_{k=1}^{\infty} C_{ik} = \emptyset.$$

Let  $\mathfrak{F}_i$  be a free filter with filter base  $\{C_{ik}\}_{k=1}^{\infty}$  and let  $\tau^*$  be the topology whose base is the collection  $\mathfrak{B} = \{\{x\} : x \in X \setminus \{x_1, x_2, \dots, x_n\}\} \cup [\bigcup_{i=1}^n \{\{x_i\} \cup C_{ik}\}_{k=1}^{\infty}]$

It is routine to check that  $\tau = \tau^*$ .

$\Leftarrow$  Let  $\{C_{ik}\}_{k=1}^{\infty}$  be the filter base for  $\mathfrak{F}_i$  for all  $i$  with  $C_{i1} \cap C_{j1} = \emptyset$  for  $i \neq j$

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, x, y \in X \setminus \{x_1, x_2, \dots, x_n\}, (x, y) \notin C_{i1} \times C_{i1} \text{ for } i = 1, 2, \dots, n \\ 0 & \text{if } x = y \\ 1 & \text{if } (x = x_i, y \notin C_{i1}) \text{ or } (y = x_i, x \notin C_{i1}) \text{ for } i = 1, 2, \dots, n \\ \frac{1}{m+1} & \text{if } x = x_i, y \in C_{i1} \text{ and } m \text{ is the least integer such that } y \notin C_{im} \\ & \text{or} \\ & y = x_i, x \in C_{i1} \text{ and } m \text{ is the least integer such that } x \notin C_{im} \\ \max\left\{\frac{1}{n+1}, \frac{1}{m+1}\right\} & \text{if } x, y \in C_{i1} \text{ and } n, m \text{ are respectively the least integers} \\ & \text{such that } x \notin C_{in}, y \notin C_{im} \end{cases}$$

Then  $d$  is a metric and induces the topology  $\tau$ .

### Proof

By induction on the number of the non-isolated points. If  $n=1$ , then this is Theorem 2.2 of [5]. If  $n=2$ , then this is Theorem 2.1. So suppose  $n \geq 3$  and the Theorem is true for all  $i$  with  $1 \leq i < n$ . Let  $d_i$  be the metric corresponding to the collection  $\{C_{ik}\}_{k=1}^{\infty}$  and the point  $x_i$  for all  $i$ .

Let  $d_*$  be the metric produced by  $d_1, d_2, \dots, d_{n-1}$  using Lemma 2.2,  $d_n$  the metric corresponding to the collection  $\{C_{nk}\}_{k=1}^{\infty}$  and the point  $x_n$  and define,

Let.

$$\mathfrak{B}_1 = \{\{x\} : \text{if } x \notin \bigcup_{i=1}^n C_{i1}\} \cup \{\{\bigcup_{i=1}^n \{x_i\} \cup C_{i1}\}\}$$

$$\mathfrak{B}_2 = \{\{x\} : \text{if } x \notin \bigcup_{i=1}^n C_{i2}\} \cup \{\{\bigcup_{i=1}^n \{x_i\} \cup C_{i2}\}\}$$

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$$\mathfrak{B}_n = \{\{x\} : \text{if } x \notin \bigcup_{i=1}^n C_{in}\} \cup \{\{\bigcup_{i=1}^n \{x_i\} \cup C_{in}\}\}$$

Then  $\mathfrak{B}_n$  is locally finite collection for all  $n$ . Let  $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$ , then  $\mathfrak{B}$  is a  $\sigma$ -locally finite collection and since  $\mathfrak{B}$  is a base for  $\tau$  by Theorem 2.2 it follows that  $X$  is a metrizable space with  $x_1, x_2, \dots, x_n$  as the non-isolated points.

### 2.9. Theorem

If  $\tau, \{C_{ik}\}_{k=1}^{\infty}$  as in last Theorem and  $d : X \times X \rightarrow \mathbb{R}$  defined by

$$d(x, y) =$$

$$\begin{cases} d_*(x, y) & \text{if } x \in X \setminus (C_{n1} \cup \{x_n\}) \text{ or } y \in X \setminus (C_{n1} \cup \{x_n\}) \\ d_n(x, y) & \text{if } x, y \in C_{n1} \cup \{x_n\} \end{cases}$$

Then by Lemma 2.2  $d$  is a metric on  $X$  with metric base the collection

$$\mathfrak{B} = \{\{x\} : x \neq x_i \text{ for all } i\} \cup [\bigcup_{i=1}^n \{\{x_i\} \cup C_{ik}\}_{k=1}^{\infty}] \text{ and clearly } \tau \text{ is generated by } \mathfrak{B}.$$

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