

On the System of Nonlinear Elliptic Equations with Nonlocal Boundary Conditions of Neumann Type

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Abstract

In this paper, we consider a system of nonlinear elliptic equations with nonlocal boundary conditions of Neumann type of the form:

$$\partial v^* u = h_{1,j} \left(x, u(\phi(x)) \right), \quad \partial v^* u = \sum_{k=1}^n f_j^k(x, u, \partial u) v_k$$

Where, $\Omega \in \mathbb{R}^n$ is a bounded domain $\partial\Omega$ and belongs to C^1 ; v_k denote the coordinates of the normal unit vector on $\partial\Omega$. Also we prove the existence of a weak solution for such problem.

المستخلص

في هذه الورقة سوف يتم دراسة منظومة المعادلات الجزيئية اللاخطية من النوع ألقصبي مع شروط حدية غير محلية من نوع نويمان، والنطاق Ω والحدود $\partial\Omega$ محدودة وتنتمي إلى الفضاء C^1 كذلك سوف يتم برهان وجود الحل الضعيف لتلك المسألة .

Introduction

The aim of this paper is to prove the existence of a weak solution $u = (u_1, u_2, \dots, u_M)$ of the system,

$$-\sum_{k=1}^n \partial_k [f_j^k(x, u, \partial u)] + f_j^0(x, u, \partial u) + g_j(x, u) = F_j \quad \text{in } \Omega, j = 1, 2, \dots, M \quad (1)$$

With nonlinear and nonlocal boundary condition of the form:

$$\partial v^* u = h_{1,j} \left(x, u(\phi(x)) \right) \quad (2)$$

Where, $\Omega \in \mathbb{R}^n$ is a bounded domain (may be unbounded),

$\partial\Omega$ is bounded and belongs to C^1 ;

$$\partial v^* u = \sum_{k=1}^n f_j^k(x, u, \partial u) v_k$$

v_k denote the coordinates of the normal unit vector on $\partial\Omega$, ϕ is C^1 ; ϕ -diffeomorphism in a neighbourhood of $\partial\Omega$ such that $\phi(\partial\Omega) \subset \bar{\Omega}$

It must be emphasized that the term $g_j(x, u(x))$ no growth restriction is imposed but it is supposed that the function g_j satisfies the sign condition $g_j(x, \eta) \eta \geq 0$. The existence of weak solution of (1) and (2) will be proved using the arguments of [1]. Weak solution of (1) and (2) will be defined as follows:

Assuming that u is a classical solution of (1) and (2) By Gauss-Ostrogrdsdskij theorem and by using an integral transformation we obtain:

$$\sum_{j=1}^M \left\{ \sum_{k=1}^n f_j^k \int_{\Omega} (x, u, \partial_1 u, \dots, \partial_n u) \partial_k v_j + \int_{\Omega} f_j^0(x, u, \partial_1 u, \dots, \partial_n u) v_j - \int_S \widetilde{h_{1,j}}(x, u) v_j (\phi^{-1}(x)) d\sigma_x + \int_{\Omega} g_j(x, u(x)) v_j \right\} \quad (3)$$

$$= \sum_{j=1}^M \int_{\Omega} F_j v_j$$

For all $v_j \in C^1(\bar{\Omega})$, with compact support, $S = \emptyset(\partial\Omega) \subset \bar{\Omega}$. Thus weak solution

$u = (u_1, u_2, \dots, u_M)$ of (1), (2) will be defined by (3).

The nonlocal linear boundary value problems have been considered e.g in [2] and [3] and the importance of nonlocal in [4]. In [4], [5] and [6] nonlocal and nonlinear value problems have been studied.

Existence Theorem

Let $X = W_p^1(\Omega) \times \dots \times W_p^1(\Omega)$. Then X is a reflexive Banach Space.

Denote by X' the dual space of X . The points

$\zeta = (\zeta_0, \zeta_1, \dots, \zeta_n) \in IR^{(n+1)M}$ ($\zeta_j = (\zeta_j^1, \zeta_j^2, \dots, \zeta_j^M) \in IR^M$) Will be written in the form $\zeta = (\eta, \zeta)$ where $\eta = \zeta_0, \zeta = (\zeta_0, \zeta_1, \dots, \zeta_n)$

Assume that:

a) Function $f_j^k: \Omega \times IR^{(n+1)M} \rightarrow IR$ satisfies Carathodory conditions, i.e. they are measurable for every fixed $\zeta \in IR^{(n+1)M}$ and continuous in ζ for a.e.x. Similarly function $h_{1,j}: S \times IR^M \rightarrow IR$ satisfies the Carathodory conditions:

b) There exists a constant $c_1 > 0$ and a function $k_1 \in L^q(\Omega)$

$\left(\frac{1}{p} + \frac{1}{q} = 1\right)$ such that $|f_j^k(x, \zeta)| \leq c_1 |\zeta|^{p-1} + k_1(x)$, $j = 1, 2, \dots, M, k = 1, 2, \dots, n$

c) For all $(\eta, \zeta), (\eta, \zeta')$ in $IR^{(n+1)M}$ with $\zeta \neq \zeta'$,

$$\sum_{j=1}^M \sum_{k=1}^n [f_j^k(x, \eta, \zeta) - f_j^k(x, \eta, \zeta')] (\zeta_j - \zeta_j') > 0$$

d) There exists a constant c_2 and a function $k_2 \in L^1(\Omega)$ such that

$$\sum_{j=1}^M \sum_{k=1}^n f_j^k(x, \zeta) \zeta_j^k \geq c_2 |\zeta|^p - k_2(x)$$

e) For any $s > 0$, there exists $g_{j,s} \in L^1(\Omega)$ such that for a.e. $x \in \Omega$

$$|g_j(x, \eta)| \leq g_{j,s}(x) \quad \text{if } |\eta| \leq s, \quad j = 1, 2, \dots, M$$

f) For any $\eta \in \mathbb{R}^M$, and for a.e. $x \in \Omega$

$$g_j(x, \eta) \eta_j \geq 0, \quad j = 1, 2, \dots, M$$

g) There exist a constants $c_3 > 0$ and a fixed function $k_3 \in L^{1+\frac{1}{\rho}}(s)$ such that

$$|h_{1,j}(x, \eta)| \leq c_3 |\eta|^\rho + k_3(x)$$

where $0 < \rho < p - 1$

Results

The main result of this paper is the following:

Theorem: Suppose that the assumptions a-g are satisfied, then for any

$$F_j \in (W_p^1(\Omega))'$$

There exists $u \in X$ such that $g_j(x, u) \in L^1(\Omega)$, $g_j(x, u)u_j \in L^1(\Omega)$, and (3) holds for all $v \in X$ Satisfying $v \in L^\infty(\Omega) \times \dots \times L^\infty(\Omega)$, $v|_{\partial\Omega} \in L^\infty(\partial\Omega) \times \dots \times L^\infty(\partial\Omega)$ and for $v = u$.

$$\begin{aligned} \text{For any } u, v \in X \text{ let } \langle T(u), v \rangle := & \sum_{j=1}^M \sum_{k=1}^n \int_{\Omega} f_j^k(x, u, \partial_1 u, \dots, \partial_n u) \partial_k v_j + \\ & + \sum_{j=1}^M \int_{\Omega} f_j^0(x, u, \partial_1 u, \dots, \partial_n u) v_j - \sum_{j=1}^M \int_s \widetilde{h_{1,j}}(x, u) v_j(\phi^{-1}(x)) d\sigma_x \end{aligned} \quad (4)$$

And for any $\mu > 0$ Let

$$g_j^\mu(x, \eta) := \begin{cases} g_j(x, \eta) & \text{if } |\eta| \leq \mu, |x| \leq \mu \\ \mu g_j(x, \eta) & \text{if } |\eta| > \mu, |x| < \mu \\ |g_j(x, \eta)| & \\ 0 & \text{if } |x| < \mu \end{cases} \quad (5)$$

Define operator S_μ by

$$\langle S_\mu(u), u \rangle := \sum_{j=1}^M \int_{\Omega} g_j^\mu(x, u) v_j \quad (6)$$

Lemma 1. Operator $T + S_\mu : X \rightarrow X'$ is pseudomontate

Proof $T + S_\mu$ is bounded. Operator T can be written in the form $T = A - B$. Where

$$\begin{aligned} \langle A(u), v \rangle := & \sum_{j=1}^M \sum_{k=1}^n \int_{\Omega} f_j^k(x, u, \partial_1 u, \dots, \partial_n u) \partial_k v_j + \\ & + \sum_{j=1}^M \int_{\Omega} f_j^0(x, u, \partial_1 u, \dots, \partial_n u) v_j \end{aligned}$$

$$\langle B(u), v \rangle := \sum_{j=1}^M \int_s \widetilde{h}_{1,s}(x, u) v_j(\emptyset^{-1}(x)) d\sigma_x$$

From conditions a-d it follows that A is pseudomonotone (See [7]). Let (u^l) be a sequence such that (u^l) converge weakly to u in X and $\limsup_{l \rightarrow \infty} \langle T(u^l), u^l - u \rangle \leq 0$

Now, by compact of trace operator there exists a subsequence $(\tilde{u}_j^l)$ of (u_j^l) such that for all j , $\tilde{u}_j^l|_{\partial\Omega}$ converges to u_j in $L^{\tilde{q}}(\partial\Omega)$. Where $\tilde{q} := \rho + 1 < p$. be condition (g) and holder is inequality

$\left(\frac{1}{p} + \frac{1}{q} = 1\right)$ we have:

$$\limsup \langle B(u^l), u^l - u \rangle = 0 \quad (7)$$

And (7) is true also for the original sequence. Further we prove that

$$B(u^l) \xrightarrow{w'} B(u) \text{ in } X'. \quad (8)$$

i.e. For All $v \in X$ $\langle B(u^l), v \rangle \rightarrow \langle B(u), v \rangle$

We have seen that there exists a subsequence $(\tilde{u}^l)$ of (u^l) such that $(\tilde{u}^l)|_{\partial\Omega}$ converges to u in $L^{\tilde{q}}(\partial\Omega)$ and converges a.e. to u on $\partial\Omega$. Thus

$$h_{1,j}(x, \tilde{u}^l) \rightarrow h_{1,j}(x, u) \text{ a.e. on } \partial\Omega$$

By Holder's inequality and the bounded of the trace operator we have:

$$\lim_{l \rightarrow \infty} \langle B(\tilde{u}^l), v \rangle = \langle B(u), v \rangle$$

and it is easy to show that the above equality is true also for the original sequence i.e. we have proved (8), thus we have shown that if (u^l) converges weakly to u in X and

$$\limsup \langle T(u^l), u^l - u \rangle \leq 0, \limsup \langle B(u^l), u^l - u \rangle = 0 \quad (9)$$

and

$$B(u^l) \xrightarrow{w'} B(u) \text{ in } X'. \quad (10)$$

From (9) it follows that

$$\limsup_{l \rightarrow \infty} \langle A(u^l), u^l - u \rangle \leq 0$$

Since A is pseudomonotone thus

$$A(u^l) \xrightarrow{w'} A(u) \text{ in } X'.$$

And by (10)

$$T(u^l) \xrightarrow{w'} T(u) \text{ in } X'.$$

By (9) we have

$$\lim_{l \rightarrow \infty} \langle T(u^l), u^l - u \rangle = 0$$

So we have shown that T is pseudomonotone operator. Now we shall prove that $T + S_\mu$ is pseudomonotone operator.

Suppose that (u^l) converges weakly to u in X and $(T + S_\mu)(u^l)$ converges weakly in X' to some y and

$$\lim_{l \rightarrow \infty} \langle (T + S_\mu)(u^l), u^l - u \rangle \leq 0 \quad (11)$$

Then by compact imbedding theorems there is a subsequence (u^{l_k}) of (u^l) such that $\lim_{l \rightarrow \infty} (u^{l_k}) = u$ a.e. in Ω and on $\partial\Omega$

Thus by Lebesgue's dominated convergence theorem

$$\lim_{l \rightarrow \infty} \|g_j^\mu(x, u^{l_k}) - g_j^\mu(x, u)\|_{L^q(\Omega)} = 0 \quad (12)$$

Where q defined by $\frac{1}{p} + \frac{1}{q} = 1$, hence $\lim_{l \rightarrow \infty} S_\mu(u^{l_k}) = S_\mu(u)$ weakly in X' and so

$$\lim_{k \rightarrow \infty} T(u^{l_k}) = y - S_\mu \quad (13)$$

weakly in X' from equality

$$\langle S_\mu(u^{l_k}), u^l - u \rangle = \langle S_\mu(u^{l_k}) - S_\mu(u), u^{l_k} - u \rangle + \langle S_\mu(u), u^{l_k} - u \rangle$$

It follows that

$$\lim_{l \rightarrow \infty} \langle S_\mu(u^{l_k}), u^{l_k} - u \rangle = 0$$

Because by (12), the boundness of $\|u^{l_k} - u\|_X, \|u_j^{l_k} - u_j\|_{L^p(\partial\Omega)}$

And Holder is inequality

$$\lim_{k \rightarrow \infty} \langle S_\mu(u^{l_k}) - S_\mu(u), u^{l_k} - u \rangle = 0 \quad (14)$$

It is not difficult to show that (14), is true also for the original sequence. Therefore (11) implies.

$$\limsup_{k \rightarrow \infty} \langle T(u^{l_k}), u^l - u \rangle \leq 0 \quad (15)$$

Since T is pseudomonotone thus by (13) and (15) we have $T(u) = y - S_\mu(u)$, i. e. $(T + S_\mu)(u) = y$

Further

$$\begin{aligned} \lim_{k \rightarrow \infty} \langle T(u^l), u^l - u \rangle &= 0 \\ \lim_{l \rightarrow \infty} \langle (T + S_\mu)(u^l), u^l - u \rangle &= 0 \end{aligned}$$

Which completes the proof of lemma 1.

lemma 2. Assume that (u^l) converges weakly to u in X and there is a constant c such that

$$\sum_{j=1}^M \int_{\Omega} g_j(x, u^l) u_j \leq c \quad (16)$$

Then

$$\mathfrak{g}_{j(x,u)} \in L^1(\Omega) , \quad \mathfrak{g}_{j(x,u)u_j} \in L^1(\Omega)$$

For all $j = 1, 2, \dots, M$ and there is a subsequence (u^{l_k}) of (u^l) such that

$$\lim_{k \rightarrow \infty} u^{l_k} = u \text{ a.e in } \Omega \text{ and on } \partial\Omega \quad (17)$$

Further

$$\lim_{k \rightarrow \infty} \left\| g_j^{l_k}(x, u^{l_k}) - g_j(x, u) \right\|_{L^1(\Omega)} = 0$$

Proof: As (u_j) converges weakly to u in X thus (by compact imbedding theorems) there exist a subsequence (u^{l_k}) of (u^l) such that

$$\lim_{k \rightarrow \infty} g_j^{l_k}(x, u^{l_k}) = g_j(x, u) \text{ for a.e } X \in \Omega \quad (18)$$

By (3),(4) and (16) and assumption (f) we have

$$\int_{\Omega} [g_j(x, u^l)] u_j^l \leq c \quad (19)$$

Therefore by (18) and (f) implies

$$g_j(x, u) u_j \in L^1(\Omega)$$

For any $\delta > 0$ we have e)

$$\left| g_j^{l_k}(x, u^{l_k}) \right| \leq g_j \delta^{-1}(x) + \delta \left| g_j^{l_k}(x, u^{l_k}) u_j^{l_k} \right|$$

This implies that $g_j^{l_k}(x, u^{l_k})$ is equiintegrable because by (19) $\int_E \left| g_j^{l_k}(x, u^{l_k}) \right| dx < \varepsilon$ if the measure of E is sufficiently small and there is a set A_ε of finite measure with

$$\int_{\Omega \times A_\varepsilon} \left| g_j^{l_k}(x, u^{l_k}) \right| < \varepsilon$$

By compact imbedding theorem and (18) this shows that

$$g_j^{l_k}(x, u^{l_k}) \rightarrow g_j(x, u) \text{ in } L^1(\Omega)$$

lemma 3 : The Operator

$T + S_\mu : X \rightarrow X'$ is coercive, i.e.

$$\lim_{\|u\| \rightarrow \infty} \frac{\langle (T + S_\mu)(u), u \rangle}{\|u\|} = +\infty$$

Proof : From f) we have

$$\int_{\Omega} g_j^\mu(x, u) u_j \geq 0$$

This implies that $\langle S_\mu(u), u \rangle \geq 0$

Thus by using conditions d) and g) we obtain

$$\frac{\langle (T+S_\mu)(u), u \rangle}{\|u\|} = \frac{\langle T(u), u \rangle}{\|u\|} + \frac{\langle S_\mu(u), u \rangle}{\|u\|} \geq \frac{\langle T(u), u \rangle}{\|u\|} \geq C_2 \|u\|_X^\rho - C_3 - C_4 \|u\|_X^{\rho+1} - C_5 \quad (20)$$

($C_2 - C_5$ are positive constants) . From this inequality and $\rho + 1 < p$ it follows that $T + S_\mu$ is coercive.

Proof of the Theorem

By Lemmas 1 and 2 the operator $T + S_j$ is bounded pseudomonotone and coercive of all $j = 1, 2, 3, \dots$, by using the well known theory of pseudo monotone operator in reflexive Banach space we obtain that for any $F \in X'$ there exists $u^l \in X$ such that

$$(T + S_j)(u^l) = F \quad (21)$$

By (20) where the constants are independent of (μ) and (21) the sequence (u^l) is bounded in X' . T is a bounded operator and so the sequence $T(u^{l_k})$ is bounded in X' . Since X is a reflexive Banach space, thus there exists a subsequence (u^{l_k}) of (u^l) and $u \in X$ such that

$$\lim(u^{l_k}) = u \text{ weakly in } X, \quad (22)$$

and $\lim T(u^{l_k}) = y$ weakly in X' For some $y \in X'$. Combining the definition of S_j with (21) we find that

$$\sum_{j=1}^M \int_{\Omega} g_j^{l_k}(x, u^{l_k}) u_j^{l_k} = \langle S_{l_k}(u^{l_k}), u^{l_k} \rangle = \langle F, u^{l_k} \rangle \leq \|F\|_{X'} \|u^{l_k}\|_X < C$$

Thus by lemma 2.

$$g_j(x, u) u_j \in L^1(\Omega), \quad g_j(x, u) \in L^1(\Omega) \quad (23)$$

And there is a subsequence (u^{l_k}) of (u^l) such that

$$\lim(u^{l_k}) = u \text{ a.e. in } \Omega \text{ and on } \partial\Omega \text{ and also} \quad (24)$$

$$\lim_{k \rightarrow \infty} \|g_j^{l_k}(x, u^{l_k}) - g_j(x, u)\|_{L^1(\Omega)} = 0 \quad (25)$$

According to (21) for any $v \in X$

$$\langle (T + S_{l_k})(u^{l_k}), v \rangle = \langle F, v \rangle \quad (26)$$

Consider in (26) a fixed $v \in X$ such that $v \in L^\infty(\Omega) \times \dots \times L^\infty(\Omega)$

And $v|_{\partial\Omega} \in L^\infty(\partial\Omega) \times \dots \times L^\infty(\partial\Omega)$

By using (23)-(26) we obtain as $k \rightarrow \infty$

$$\langle y, v \rangle + \sum_{j=1}^M \int_{\Omega} g_j(x, u) v_j = \langle F, v \rangle \quad (27)$$

Now, we shall prove that $y = T(u)$.

Since T is pseudomonotone, thus, it is sufficient to prove the following inequalities

$$\lim_{k \rightarrow \infty} \sup \langle T(u^{l_k}), u^{l_k} - u \rangle \leq 0$$

We have,

$$\langle T(u^{l_k}), u^{l_k} - u \rangle = \langle T(u^{l_k}), u^{l_k} \rangle - \langle T(u^{l_k}), u \rangle$$

And so by (22) and (26) and lemma 2.

$$\begin{aligned} \lim_{k \rightarrow \infty} \sup \langle T(u^{l_k}), u^{l_k} - u \rangle &= \lim_{k \rightarrow \infty} \sup \langle F - S^{l_k}(u^{l_k}), u^{l_k} \rangle - \langle y, v \rangle - \\ &\leq \langle F - y, v \rangle - \lim_{k \rightarrow \infty} \inf \left\{ \sum_{j=1}^M \int_{\Omega} g_j^{l_k}(x, u^{l_k}) v_j \right\} \leq \langle F - y, v \rangle - \sum_{j=1}^M \int_{\Omega} g_j(x, u) v_j \end{aligned}$$

Thus for any $w \in X \cap L^\infty(\Omega) \times \dots \times L^\infty(\Omega)$ by using (27).

$$\lim_{k \rightarrow \infty} \sup \langle T(u^{l_k}), u^{l_k} - u \rangle \leq \langle F - y, u - w \rangle + \sum_{j=1}^M \int_{\Omega} [g_j(x, u)] (u_j - u) \quad (28)$$

Since $\partial\Omega$ is bounded and continuously differentiable, thus $u \in X$ can be extended to IR^n such that we obtain

$$u \in w_p^1(IR^n) \times \dots \times w_p^1(IR^n)$$

We know that there is a subsequence (w_j^l) and $w_p^1(IR^n) \cap L^\infty(IR^n)$ such that (w_j^l) in $w_p^1(IR^n) \cap L^\infty(IR^n)$ and such that (w_j^l) converges to (u_j) in $w_p^1(IR^n)$ and a.e. in IR^n , further

$$|w_j^l(x)| \leq |u_j(x)| \text{ a.e. in } IR, j = 1, 2, \dots, M \quad (29)$$

consequently for the trace of (w_j^l) and u_j . (29)

$$|w_j^l|_{\partial\Omega}(x) \leq |u_j|_{\partial\Omega}(x) \text{ in a.e. } X \in \partial\Omega. \quad (30)$$

By (29) and (30) we have

$$\langle F - y, u - w^l \rangle \rightarrow 0 \text{ and } \int_{\Omega} [g_j(x, u)] w_j^l dx \rightarrow \int_{\Omega} [g_j(x, u)] u_j dx$$

Since $g(x, u)u \in L^1(\Omega)$. Thus from (28) it follows that

$$\lim_{k \rightarrow \infty} \sup \langle T(u^{l_k}), u^{l_k} - u \rangle \leq 0$$

Consequently, $y = T(u)$, and

$$\langle T(u^{l_k}), u^{l_k} - u \rangle \rightarrow 0$$

Therefore from (27) we obtain (3) for all $v \in w_p^1(\Omega) \times \dots \times w_p^1(\Omega)$

with $v \in L^\infty(\Omega) \times \dots \times L^\infty(\Omega)$

$v|_{\partial\Omega} \in L^\infty(\partial\Omega) \times \dots \times L^\infty(\partial\Omega)$.

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